1. a) Prove that for any ring R, if $U, V \leq R_R$ such that R_R/U is semisimple and V is nilpotent then $V \leq J(R) \leq U$.
b) Prove that $\mathbb{Z}/J(\mathbb{Z})$ is not semisimple.
c) Let $R = \left\{ \frac{a}{b} \;\middle|\; a, b \in \mathbb{Z}, b \text{ is odd} \right\}$ and $I = 2R$. Prove that $I = J(R)$, and I is not nilpotent.

2. Prove that the following statements are equivalent for a ring R.
a) R is semisimple.
b) Every R-module is projective.
c) Every R-module is injective.

3. Let $e_1, \dots, e_n$ be a complete set of orthogonal idempotents in A, and $M \in \text{Mod-}A$. Take the decomposition $M = M_1 \oplus \dots M_n$ of M into a direct sum of subspaces $M_i = Me_i$. Show that the elements of e_iAe_j act as linear maps from M_i to M_j, and the action of e_iAe_j $(i, j = 1, \dots, n)$ determines the action of A on M.

4. Consider the graph algebra $K\Gamma/I$, where Γ : $\overset{1}{\bullet} \xrightarrow{\alpha} \overset{2}{\bullet} \circlearrowright \beta$ and $I = (\alpha\beta^2, \beta^3)$. Let $M = M_1 \oplus M_2$ be a vector space such that $\dim_K M_1 = \dim_K M_2 = 2$, and fix a basis $\mathcal{B} = \{ b_1, b_2 \}$ in M_1 and $\mathcal{C} = \{ c_1, c_2 \}$ in M_2. We define the action of A as $xe_i = x$ if $x \in M_i$ and 0 if $x \in M_j$ $(j \neq i)$ and the matrix of $M_1 \xrightarrow{\alpha} M_2$ in $(\mathcal{B}, \mathcal{C})$ is $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and the matrix of $M_2 \xrightarrow{\beta} M_2$ in $(\mathcal{C}, \mathcal{C})$ is $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.
a) Show that with the natural extension of the action of A, M becomes an A-module.
b) Determine the Loewy diagram of M, using the basis $\mathcal{B} \cup \mathcal{C}$.
c) Find the Loewy diagram of the submodule U of M generated by $b_1 - b_2$, and the Loewy diagram of the factor module M/U.

5. Give a basis and the Loewy diagram of the indecomposable direct summands of A_A if $A = K\Gamma/I$, where Γ: $\overset{1}{\bullet} \underset{\beta}{\overset{\alpha}{\rightleftarrows}} \overset{2}{\bullet} \circlearrowright \gamma$ and
a) $I = (\alpha\gamma, \gamma^2, \gamma\beta, \alpha\beta\alpha, \beta\alpha\beta)$; b) $I = (\alpha\gamma^2, \gamma^2 - \beta\alpha, \alpha\beta)$.

6. Is there a graph algebra $K\Gamma/I$ such that the Loewy diagram of the regular module is the one shown below? If yes, give the graph Γ and a generator system of the ideal I.
a) $\begin{smallmatrix}1\\2\\2\\2\end{smallmatrix} \oplus \begin{smallmatrix}2\\2\end{smallmatrix}$ b) $\begin{smallmatrix} & 1 & \\ 1 & & 2 \\ & 2 & \end{smallmatrix} \oplus \begin{smallmatrix} & 2 & \\ 1 & & 2 \end{smallmatrix}$ c) $\begin{smallmatrix}1\\2\end{smallmatrix} \oplus \begin{smallmatrix}2\\3\end{smallmatrix} \oplus 3$

7. Let $A = K\Gamma/I$ be a graph algebra with the Loewy diagram $A_A = \begin{smallmatrix}1\\2\\2\end{smallmatrix} \oplus \begin{smallmatrix}2\\2\end{smallmatrix}$ from problem 4. Determine all the submodules and their factors for the module $\begin{smallmatrix}1 & & 2\\ & 2 & \\ & 2 & \end{smallmatrix}$.

8. What is the dimension of the vector spaces $\mathrm{Hom}(\begin{smallmatrix} & 2 & \\ 1 & & 2 \end{smallmatrix}, \begin{smallmatrix} 1 & & 2 \\ & 2 & \end{smallmatrix})$, and $\mathrm{Hom}(\begin{smallmatrix} 1 & & 2 \\ & 2 & \end{smallmatrix}, \begin{smallmatrix} & 2 & \\ 1 & & 2 \end{smallmatrix})$ for modules over the algebra of problem 5.a).

Hf1. Determine the Jacobson radical of the ring R of 3×3 upper triangular matrices over $\mathbb{Z}_2$, and the radical of the R-module M of all 3×3 matrices over $\mathbb{Z}_2$.

Hf2. What is the Loewy diagram of the regular module of $A = K\Gamma/I$ if

$$\Gamma : 1 \underset{\beta}{\overset{\alpha}{\rightleftarrows}} 2 \underset{\delta}{\overset{\gamma}{\rightleftarrows}} 3, \qquad I = (\alpha\gamma, \beta\alpha - \gamma\delta, \delta\beta)?$$