1. Theorem: Let $0 \to X \to Y \to Z \to 0$ be an Auslander–Reiten sequence.

1) There is an irreducible morphism $Z' \to Z$ if and only if $Z' \overset{\oplus}{\leq} Y$;
2) There is an irreducible morphism $X \to X'$ if and only if $X' \overset{\oplus}{\leq} Y$.

Prove the 'only if ' direction in both statements.

2. Let $A_A = P_1 \oplus P_2 \oplus \ldots \oplus P_n$ be a decomposition into indecomposable projective modules. We define a graph on $\mathcal{P} = \{ P_1, \ldots, P_n \}$ so that P_i and P_j are connected with an edge if and only if $\mathrm{Hom}(P_i, P_j)$ or $\mathrm{Hom}(P_j, P_i)$ is nonzero. Let $\mathcal{K}_1, \ldots, \mathcal{K}_t$ be the connected components of this graph. Prove that every $R_j := \oplus \{ P_i \mid P_i \in \mathcal{K}_j \}$ is an idecomposable ideal of A, so A is connected if and only if the graph on $\mathcal{P}$ is connected. In particular, a graph algebra $K\Gamma/I$ is connected if and only if Γ is connected.

3. Let A be a graph algebra such that $A_A = \begin{smallmatrix} 1 \\ 3 \\ 1 \end{smallmatrix} \oplus \begin{smallmatrix} 2 \\ 1 \end{smallmatrix} \oplus \begin{smallmatrix} 3 \\ 1 \end{smallmatrix}$. Determine the Auslander–Reiten translate of the simple modules.

4. Determine the Auslander–Reiten graph of the following graph algebras.

a) $A = K\Gamma$, where $\Gamma : 1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3 \xleftarrow{\gamma} 4$.
b) $A_A = \begin{smallmatrix} 1 \\ 2 \end{smallmatrix} \oplus \begin{smallmatrix} 2 \\ 2 \end{smallmatrix}$

5. Prove that $Z(M_n(R)) = Z(R)I_n$, where I_n denotes the $n \times n$ identity matrix.

6. Let $S = eR \overset{\oplus}{\leq} R$ be a simple module generated by the idempotent element e. Prove that $\mathrm{End}\, S \cong eRe$. In particular, if R is a full matrix ring over a division ring D then $\mathrm{End}\, S \cong D$.

7. Find the irreducible representations of C_3 over an arbitrary field K. Determine the submodules of KC_3 when $\mathrm{char}\, K = 3$.

8. Find the irreducible representations of $C_2 \times C_2$ over an arbitrary field K.

9. Prove that $J(KG) = \left\{ \sum_{g \in G} \lambda_g g \;\middle|\; \sum_{g \in G} \lambda_g = 0 \right\}$ if G is a finite p-group and $\mathrm{char}\, K = p$. How many nonisomorphic simple modules exist in mod-KG?

HW1. Consider the graph algebra with Loewy diagram $A_A = \begin{smallmatrix} & 1 & \\ 2 & & 3 \\ 1 & & \end{smallmatrix} \oplus \begin{smallmatrix} 2 \\ 1 \\ 2 \end{smallmatrix} \oplus \begin{smallmatrix} 3 \end{smallmatrix}$. Calculate the AR translate of the module $\begin{smallmatrix} 1 \\ 2 \\ 1 \end{smallmatrix}$.

HW2. Prove that the nonzero morphism $\begin{smallmatrix} 2 \\ 1 \end{smallmatrix} \to \begin{smallmatrix} 1 \\ 2 \\ 1 \end{smallmatrix}$ is not an irreducible morphism. (Use the result of HW1, and show that the first module cannot be a direct summand of the middle term of the AR sequence, or give a proper decomposition of the morphism, and prove that it is proper.)