1. Prove the orbit-counting lemma: if $\varphi : G \to S_\Omega$ is an action of the group G on a set Ω, then the number of orbits of G in Ω is

$$\frac{1}{|G|}\sum_{g\in G} |\operatorname{Fix}(g)|.$$

2. Prove that 1_G is always a summand in the decomposition of any permutation character into irreducibles. What is the coefficient of 1_G?

3. Prove that for the permutation character χ of a 2-transitive group action, $\chi - 1_G$ is always irreducible.

4. Let $H \le K \le G$, and let φ be a class function of H. Show that $(\varphi^K)^G = \varphi^G$.

5. What is the induced character to A_4 of

a) the trivial character of the Klein group $\langle (..)(..)\rangle$;

b) a nontrivial character of the cyclic group $\langle (123)\rangle$?

6. a) Consider the action of S_4 on the partitions of $\{1,2,3,4\}$ to subsets of the following sizes: 4, $3+1$, $2+2$, $2+1+1$, $1+1+1+1$, where the order of the subsets matters but the order of the elements within each subset does not. Calculate the permutation characters corresponding to these group actions, and write them as sums of irreducible characters.

b) Use the method of part a) to determine the character table of S_5: calculate the permutation characters obtained from the group action on the partitions to sizes 5, $4+1$, $3+2$ and $3+1+1$ (the subsets are in decreasing order, and the partition types are taken in lexicographic order), and find a new irreducible character from each by subtracting the already known irreducible components. Instead of continuing with the remaining partitions, complete the table by multiplying the irreducible characters by the alternating linear character.

c) Restrict the irreducible characters of S_5 to A_5. Which of the restrictions are irreducible. Complete the character table of A_5 by taking the induced character of a nontrivial character of $\langle (12345)\rangle$ to A_5.

HW1. By the character table of S_4

S_4	1	(..)(..)	(...)	(..)	(....)
χ_1	1	1	1	1	1
χ_2	1	1	1	-1	-1
χ_3	2	2	-1	0	0
χ_4	3	-1	0	1	-1
χ_5	3	-1	0	-1	1

write the product of the two irreducible characters of degree 3 as a sum of irreducible characters.

HW2. Determine the character of S_4 induced from the trivial character of the 2-Sylow subgroup $\langle (1234), (12)(34)\rangle$, and write it as a sum of irreducible characters.