1. Let X be a representation of G and χ the corresponding character. Prove that
a) $Z(\chi) = \{\, g \in G \mid X(g) = \varepsilon I \text{ for some } \varepsilon \in \mathbb{C} \,\}$;
b) $Z(\chi) \lhd G$.

2. Prove that the conjugate of any character is a character and the conjugate of an irreducible character is an irreducible character.

5. a) Prove that $\sum\limits_{\chi \in \operatorname{Irr} G} \sum\limits_{g \in G} \chi(g)^2$ is an integer divisible by $|G|$.
b) Prove that the number of self-inverse conjugacy classes is the same as the number of real irreducible characters.

4. Let χ be a character of the group G. Consider the map $\det \chi : G \to \mathbb{C}^\times$, where $(\det \chi)(g) = \det X(g)$ and X is a representation for the character χ. Prove that $\det \chi$ is a well-defined linear character.

5. Prove that a simple group cannot have an irreducible character of degree 2. (Use Problem 5.)

A conjugacy class of even permutations of S_n is either also a conjucacy class of A_n or it splits into two conjugacy classes of equal size. The conjugacy class splits if and only if the cyclic decomposition contains only cycles of odd length and it contains at most one cycle of any length (including the 1-cylces).

6. a) List the conjugacy classes of S_6 and determine their sizes.
b) List the conjugacy classes of A_8 and determine the sizes of those that are not complete conjugacy classes in S_8.

7. a) Consider the dihedral group D_4 to be a subgroup of S_4 by the action on a square with corners $1, 2, 3, 4$. Determine the permutation character of D_4 defined by its action on the two-element subsets of $\{\, 1, 2, 3, 4 \,\}$.
b) Determine the permutation character of D_4 defined by the action on all 3-colourings of the corners of the square.
c) Determine the permutation character of S_6 defined by the action on the partitions of type $3 + 3$.
d) Determine the scalar product of the characters in a), b) and c) with the trivial character (that is, the number of orbits of the given group action), and the scalar square of the character. What can we say about the irreducible components?

HW1. Determine the value of $\sum\limits_{\chi_i, \chi_j \in \operatorname{Irr} G} \sum\limits_{g \in G} \chi_i(1)\chi_j(1)\chi_i(g)\chi_j(g)$.

HW2. Determine the permutation character χ of S_4 defined by the action on the partitions of type $4 + 2$. Calculate $[\chi, \chi]$. How many irreducible components can χ have?