1. *Let $M \in$ Mod-R be a right R-module, B a left ideal and J a right ideal of R, $a \in M$, and U, V submodules in M. Which of the following are necessarily submodules of M? (For the sets X, Y, the sum means $X + Y = \{\, x + y \,|\, x \in X, y \in Y \,\}$, the product $XY = \{\, \sum_i x_i y_i \,|\, x_i \in X,\ y_i \in Y\ \forall i \,\}$, while the annihilator Ann_M denotes the set of elements of M whose product with all the elements of the given subset of the ring is 0.)*

a) aR *b)* aB *c)* aJ *d)* $U \cap V$ *e)* $U \cup V$
f) $U + V$ *g)* $\mathrm{Ann}_M(B)$ *h)* $\mathrm{Ann}_M(J)$ *i)* UB *j)* UJ.

Solution: Submodules: aR, aJ, $U \cap V$, $U + V$, $\mathrm{Ann}_M(B)$, UJ.
Counterexamples: Let $R = \mathbb{R}^{n\times n}$, and

$$M = R_R, \quad a = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, U = \left\{ \begin{bmatrix} * & * \\ 0 & 0 \end{bmatrix} \right\}, \quad V = J = \left\{ \begin{bmatrix} 0 & 0 \\ * & * \end{bmatrix} \right\}, \quad B = \left\{ \begin{bmatrix} * & 0 \\ * & 0 \end{bmatrix} \right\}.$$

For these, $aB = \left\{ \begin{bmatrix} * & 0 \\ 0 & 0 \end{bmatrix} \right\}$ is not a right module because $aBa = \left\{ \begin{bmatrix} 0 & * \\ 0 & 0 \end{bmatrix} \right\} \not\subseteq aB$ (where the a on the right is meant as an element of R);
$U \cup V$ is not a right module, since it is not closed under addition, e.g. $I \in U + V$ but $I \notin U \cup V$;
$\mathrm{Ann}_M(J) = B$ is not a right module, e.g. $Ba = \left\{ \begin{bmatrix} 0 & * \\ 0 & * \end{bmatrix} \right\} \not\subseteq B$;
$UB = \left\{ \begin{bmatrix} * & 0 \\ 0 & 0 \end{bmatrix} \right\}$ is not a right module, as we have seen above.

2. *What can be the additive group of a (unary) module over $\mathbb{Z}$, $\mathbb{Z}_3$ or $\mathbb{Z}_6$? Determine the number of 12-element modules over each ring up to isomorphism.*

Solution: Every abelian group is a $\mathbb{Z}$-module.

$\mathbb{Z}_3$ is a field, so the $\mathbb{Z}_3$-modules are vector spaces. Since every vector space is a direct sum of 1-dimensional subspaces, the additive groups of $\mathbb{Z}_3$-modules are direct sums of cyclic groups of order 3 (i.e. they are elementary abelian 3-groups).

Finally, if M is a module over $\mathbb{Z}_6$, then every element of the additive group of M has order 1, 2, 3 or 6. Let M_2 be the set of elements of order 2 or 1, and M_3 the set of elements of order 3 or 1. These are clearly subgroups of $(M, +)$ (actually, they are submodules of M), such that $M_2 \cap M_3 = 0$. Furthermore, any element $m \in M$ can be written as $m = 7m = (3m) + (4m) \in M_2 + M_3$, so $M = M_2 \oplus M_3$, where by the previous case (and the similar case of $\mathbb{Z}_2$-modules) M_2 is an elementary abelian 2-group, while M_3 is an elementary abelian 3-group.

If M is a 12-element module over $\mathbb{Z}$, $\mathbb{Z}_3$ or $\mathbb{Z}_6$, then M as an abelian group is isomorphic to $\mathbb{Z}_4 \oplus \mathbb{Z}_3$ or $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3$ by the fundamental theorem of finite abelian groups. None of these is a $\mathbb{Z}_3$-module (they have elements of order 2 and 6), and only the second is a $\mathbb{Z}_6$-module (the other has elements of order 4, which is not a divisor of 6), so there are 2 $\mathbb{Z}$-modules, 0 $\mathbb{Z}_3$-module and 1 $\mathbb{Z}_6$-module of 12 elements, up to isomorphism.

3. *a) Let $1 \in S \leq R$. Prove that every R-module is also an S-module, but the converse is not true.*
b) Suppose that $I \lhd R$. What is the connection between the modules over R and the modules over R/I?

Solution: a) It is clear that any R-module is closed under multiplication by elements of S. On the other hand, the action of the subring on an S-module may not be extended to an action of R. For example, the abelian group $\mathbb{Z}_2$ is a module over $\mathbb{Z}$ but it cannot be a module over $\mathbb{R} \geq \mathbb{Z}$, since a nontrivial vector space over $\mathbb{R}$ must have infinitely many elements.

b) Every R/I-module is also an R-module (with the multiplication $mr := m(r + I)$). Conversely, an R-module M is an R/I-module if and only if $MI = 0$ (in that case the multiplication $m(r + I) = mr$ is well-defined, and the validity of the axioms follows from the properties of the R-module).

4. *Let $V = \mathbb{R}^n$ be an n-dimensional vector space over $\mathbb{R}$. Find a subring S of the ring of $n\times n$ matrices such that the only nontrivial S-submodule with respect to the usual vector-matrix multiplication is the following.*

a) $U = \{(x_1, \ldots, x_n) \in V \mid x_1 + \ldots + x_n = 0\}$
b) $U = \{(x, \ldots, x) \in V \mid x \in \mathbb{R}\}$

Solution: For a subspace $U \leq V$, consider the subring $S = \{\varphi \in \operatorname{End} V \mid U\varphi \leq U\} \leq \operatorname{End} V \cong \mathbb{R}^{n\times n}$. We are going to show that U is the only proper S-invariant subspace of V. If $W \not\leq U$, and $W < V$, then for the vectors $w \in W \setminus U$ and $v \in V \setminus W$ there exists a linear transformation $\varphi \in \operatorname{End} V$ such that $w\varphi = v$ and $U\varphi = 0$, so $\varphi \in S$ but $W\varphi \not\leq W$. On the other hand, if $0 < W < U$, then for $u \in U \setminus W$, $w \in W \setminus \{0\}$ there exists a map $\varphi \in S$, such that $\varphi : w \mapsto u$ (and φ is 0 on a the rest of a basis of U, which contains w), so $W\varphi \not\leq W$.

In part a), this S consists of the matrices for which the sum of every row is the same, while in part b), the same condition holds for the columns.

5. *Let A, B, C be submodules of M such that $A \geq C$. Prove that $A\cap(B+C) = (A\cap B)+C$.*

Solution:
$A \cap (B + C) \geq (A \cap B) + C$:
$A \geq A \cap B$ and $A \geq C$ implies that $A \geq (A \cap B) + C$.
$B + C \geq B \geq A \cap B$ and $B + C \geq C$ implies that $B + C \geq (A \cap B) + C$.
So $A \cap (B + C) \geq (A \cap B) + C$.
$A \cap (B + C) \leq (A \cap B) + C$:
For an element $a = b + c \in A \cap (B + C)$ (where $a \in A$, $b \in B$, $c \in C$), $b = a - c \in A$, since $C \leq A$, so $b \in A \cap B$, consequently, $b + c \in (A \cap B) + C$.

6. *Suppose that $N \leq M \in$ Mod-R. Prove that M has a maximal submodule U such that $N \cap U = 0$, and that for such a module U, the intersection of $N \oplus U$ with any nonzero submodule of M is nonzero. Give an example among abelian groups to show that $N \oplus U$ is not necessarily the whole M.*

Solution: The set $\mathcal{U}$ of submodules disjoint from N satisfies the conditions of Zorn's lemma: if we take a chain of such modules then its union is also in $\mathcal{U}$. So $\mathcal{U}$ has a maximal element, let this be U.

If for some submodule $V \leq M$, we have $V \cap (N + U) = 0$, then $N \cap (U + V) = 0$ ($n = u+v \Rightarrow v = n-u \in V\cap(N+U) = 0 \Rightarrow v = 0 \Rightarrow n = u \in N\cap U = 0 \Rightarrow n = u = 0$). So by the maximality of U, we get that $U+V = U$, i.e. $V \leq U$, thus $V = V\cap(N+U) = 0$.

In the abelian group $\mathbb{Z}_4$, the zero module is the only submodule which is disjoint from $\langle 2\rangle$, so this is the maximal disjoint submodule. But $0 + \langle 2\rangle \neq \mathbb{Z}_4$.

7. *Which of the following classes of modules have the property that every module can be written as a direct sum of cyclic, or of simple modules?*
a) vector spaces
b) modules over a division ring
c) finite abelian groups
d) abelian groups
e) modules over $\mathbb{Z}_n$
f) modules over $K[x]$, where K is a field

Solution: a) Every vector space V has a basis: $\mathcal{B} = \{\, b_i \mid i \in I\,\}$, and this gives a decomposition $V = \bigoplus_{i\in I} \langle b_i\rangle$ into simple modules.

b) The same as in a).

c) According to the fundamental theorem of finite abelian groups, every finite abelian group can be written as a direct sum of cyclic modules. But they usually cannot be decomposed into a direct sum of simple module (e.g. $\mathbb{Z}_4$).

d) $(\mathbb{Q}, +)$ as a $\mathbb{Z}$-module cannot be decomposed into the direct sum of more than one nonzero modules, since it has no disjoint submodules ($0 \neq ac \in \frac{a}{b}\mathbb{Z} \cap \frac{c}{d}\mathbb{Z}$ if $\frac{a}{b},\ \frac{c}{d} \neq 0$). On the other hand, $\mathbb{Q}$ is not a cyclic $\mathbb{Z}$-module, since in $\frac{a}{b}\mathbb{Z}$ every denominator (in the simplified form of the rational number) is a divisor of b. So $\mathbb{Q}$ cannot be decomposed into a direct sum of cyclic modules.

e) The order of every element of a $\mathbb{Z}_n$-module is a divisor of n. By Prüfer's theorem every abelian groups of finite exponent can be written as a directs sum of cyclic subgroups. But these, in general, cannot be decomposed into a direct sum of simple modules.

f) We can use an argument very similar to that in d) to show that the field $K(x)$ of rational functions as a $K[x]$-module cannot be decomposed into a direct sum of cyclic submodules.

HW1. *Consider the following set M as a right module over the ring R.*

$$M = \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \mid a, b \in K \right\} \quad \text{és} \quad R = \left\{ \begin{bmatrix} x & y \\ 0 & x \end{bmatrix} \mid x, y \in K \right\},$$

where K is a field. Prove that every submodule of M is also a K-subspace. Determine all the 1-dimensional submodules of M. How many such submodules exist if $K = \mathbb{Z}_5$?

HW2. *Prove that the group algebra of a nontrivial, not necessarily finite group cannot be a division algebra. (Hint: Show that $\left\{ \sum_{g\in G} \lambda_g g \in KG \;\middle|\; \sum_{g\in G} \lambda_g = 0 \right\}$ is an ideal in KG.)*