1. *Let $\varphi \in \mathrm{Hom}_R(M, N)$. Prove that*
a) φ is surjective $\Leftrightarrow$ ($\varphi\alpha = \varphi\beta \;\Rightarrow\; \alpha = \beta$) for all $\alpha, \beta \in \mathrm{Hom}_R(N, L)$;
a) φ is injective $\Leftrightarrow$ ($\alpha\varphi = \beta\varphi \;\Rightarrow\; \alpha = \beta$) for all $\alpha, \beta \in \mathrm{Hom}_R(L, M)$;

Solution: a) If φ is surjective, and $\varphi\alpha = \varphi\beta$ then for every $n \in N$, there is an $m \in M$ such that $n = m\varphi$, so $n\alpha = m\varphi\alpha = m\varphi\beta = n\beta$, proving that $\alpha = \beta$.
If φ is not surjective, let $U = \mathrm{Im}\,\varphi < N$, and take $\alpha, \beta : N \to N/U$ such that α is the map $n \mapsto n + U$, while $\beta = 0$. Then $\varphi\alpha = \varphi\beta = 0$ but $\alpha \neq \beta$.
b) If φ is injective, and $\alpha\varphi = \beta\varphi$ then for every $\ell \in L$, we have $\ell\alpha\varphi = \ell\alpha\varphi$, which implies by the injectivity of φ that $\ell\alpha = \ell\beta$. So $\alpha = \beta$.
If φ is not injective, and $U = \mathrm{Ker}\,\varphi \neq 0$ then let α be the natural embedding of U into M, while $\beta : U \to M$ is taken to be the zero map. Then $\alpha\varphi = \beta\varphi = 0$ but $\alpha \neq \beta$.

2. *Let $X, Y, Z \in \text{Mod-}R$. Prove that $Y \cong X \oplus Z \Leftrightarrow$*
$\exists\ X \underset{\beta}{\overset{\alpha}{\rightleftarrows}} Y \underset{\delta}{\overset{\gamma}{\rightleftarrows}} Z$ such that $\alpha\gamma = 0$, $\delta\beta = 0$, $\alpha\beta = \mathrm{id}_X$, $\delta\gamma = \mathrm{id}_Z$, $\beta\alpha + \gamma\delta = \mathrm{id}_Y$.

Solution: If $Y = X \oplus Z$, then the projections π_1 and π_2 on the first and the second component, respectively, and the corresponding embeddings ι_1 and ι_2 satisfy the properties $\iota_1\pi_2 = 0$, $\iota_2\pi_1 = 0$, $\iota_1\pi_1 = \mathrm{id}_X$, $\iota_2\pi_2 = \mathrm{id}_Z$ and $\pi_1\iota_1 + \pi_2\iota_2 = \mathrm{id}_Y$. If Y is only isomorphic to $X \oplus Z$, say $\varphi : Y \to X \oplus Z$ is an isomorphism, then $\alpha = \iota_1\varphi^{-1}$, $\delta = \iota_2\varphi^{-1}$, $\beta = \varphi\pi_1$ and $\gamma = \varphi\pi_2$ satisfy the given equalities.

Conversely, suppose that $\alpha, \beta, \gamma, \delta$ satisfy the given equalities. Let $U = \mathrm{Im}\,\alpha$ and $V = \mathrm{Im}\,\delta$. Then $\alpha\beta = \mathrm{id}_X$ implies that α is injective, and $\delta\gamma = \mathrm{id}_Z$ implies that δ is injective, so $X \cong U$ and $Z \cong V$. $U \cap V = 0$, because for a $y \in U \cap V$, $y = x\alpha = z\delta$ for some $x \in X$ and $z \in Z$, so $x = x\alpha\beta = z\delta\beta = z0 = 0 \Rightarrow y = 0\alpha = 0$. Finally, $U + V = Y$, since for any $y \in Y$, we have $y = y\beta\alpha + y\gamma\delta \in \mathrm{Im}\,\alpha + \mathrm{Im}\,\delta$. Thus $Y = U \oplus V \cong X \oplus Z$.

3. *Let $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \overset{\alpha}{\twoheadrightarrow} \mathbb{Z}_2 \overset{\beta}{\longleftarrow} \mathbb{Z} \oplus \mathbb{Z}$ such that $(x, y)\alpha = x + y$ and $(x, y)\beta = x$. Complete this into a commutative diagram with $\mathbb{Z} \oplus \mathbb{Z} \overset{\gamma}{\longrightarrow} \mathbb{Z}_2 \oplus \mathbb{Z}_2$ in two ways, so that γ is is surjective in the first, but not surjective in the second.*

Solution: We only have to choose the image of the free generators $(1, 0)$ and $(0, 1)$ of $\mathbb{Z} \oplus \mathbb{Z}$ by the map γ. $(1, 0)\beta = 1$ and an inverse image of 1 by α can be $(1, 0)$ or $(0, 1)$. Let us choose $\gamma : (1, 0) \mapsto (1, 0)$. For the other generator, $(0, 1)\beta = 0$, and its inverse image by β can be $(0, 0)$ or $(1, 1)$. If we choose $\gamma : (0, 1) \mapsto (0, 0)$, and extend it to a homomorphism, then we get a γ, which is not surjective, but if we choose $\gamma : (0, 1) \mapsto (1, 1)$, then $\mathrm{Im}\,\gamma = \langle (1, 0), (1, 1) \rangle = \mathbb{Z}_2 \oplus \mathbb{Z}_2$, so γ will be surjective.

4. *Determine all the (finite) projective modules over $\mathbb{Z}_n$.*

Solution: Every finite abelian group (and by Prüfer's theorem, also every infinite abelian group of bounded exponent) can be written as a direct sum of cyclic groups of prime-power order. Since direct sums and direct summands of projective modules are projective, it is enough to determine which cyclic groups of prime-power order are projective over $\mathbb{Z}_n$. If P is a cyclic projective module then it is a homomorphic image of $R = \mathbb{Z}_n$, and by the projectivity, it must be a direct summand of $\mathbb{Z}_n$. For $n = p_1^{a_1} \cdots p_r^{a_r}$ (where $p_1, \ldots, p_r$ are different primes), $\mathbb{Z}_n = \mathbb{Z}_{p_1^{a_1}} \oplus \ldots \oplus \mathbb{Z}_{p_r^{a_r}}$, and this decomposition is unique by the fundamental theorem of finite abelian groups, so P must be isomorphic to one of these summands. This means that the projective modules are exactly those whose canonical cyclic decomposition contains only cyclic groups of order $p_1^{a_1}, \dots, p_r^{a_r}$.

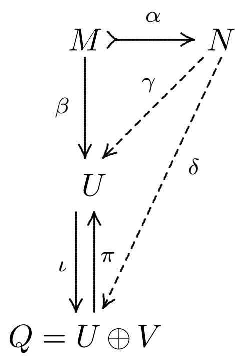
5. *Prove the following two properties about injective modules, similarly to the proof of the corresponding properties of projective modules.*

a) Every direct summand of an injective module is injective.

b) Any direct product of injective modules is injective.

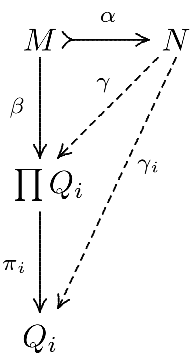
Solution:

a)



Let $Q = U \oplus V$ be an injective module, π the projection of Q onto U and ι the embedding of U into Q. Furthermore, let $\alpha : M \to N$ be an injective homomorphism, and $\beta : M \to U$. By the injectivity of Q, there is a homomorphism $\delta : N \to Q$ such that $\beta\iota = \alpha\delta$. Then for $\gamma = \delta\pi$, we have $\alpha\gamma = \alpha\delta\pi = \beta\iota\pi = \beta\,\mathrm{id}_u = \beta$, which proves the injectivity of U.

b)



Let Q_i ($i \in I$) be injective modules, $\alpha : M \to N$ an injective homomorphism, $\beta : M \to \prod Q_i$, and π_i the projection of $\prod Q_i$ on the i'th component. By the injectivity of the Q_i there exists a homomorphism $\gamma_i : N \to Q_i$ for every i such that $\alpha\psi_i = \beta\pi_i$. We define $\gamma : N \to \prod_{i\in I} Q_i$: for any $n \in N$, let $n\gamma = (n\gamma_i)_{i\in I} \in \prod_{i\in I} Q_i$. This is clearly a module homomorphism, and for any $m \in M$, we have $m\alpha\gamma = (m\alpha\gamma_i)_{i\in I} = (m\beta\pi_i)_{i\in I} = m\beta$, so $\alpha\gamma = \beta$.

6. *Prove that $\mathbb{Q}$ is not projective as a $\mathbb{Z}$-module.*

Solution: $\mathbb{Q}$ is divisible, i.e. for every $x \in \mathbb{Q}$ and $0 \neq n \in \mathbb{Z}$ there is an element $y \in \mathbb{Q}$ such that $yn = x$. On the other hand, in a direct sum of regular $\mathbb{Z}$-modules no nonzero element is divisible: if $0 \neq a = (a_i)_{i\in I} \in F = \oplus_{i\in I}\mathbb{Z}$, and $n > \max_i |a_i|$, then $a \neq nb$ for any $b \in F$. So $\mathbb{Q}$ cannot be a submodule of a free module, consequently, $\mathbb{Q}$ cannot be projective.

7*. *Prove that every subgroup of a free abelian group is free. (Hint: Let* $G = \bigoplus_{\alpha<\kappa} \langle g_\alpha \rangle$*, where* κ *is a cardinality, and* $G_\alpha = \bigoplus_{\beta<\alpha} \langle g_\beta \rangle$ *for every ordinal number* $\alpha < \kappa$*. For a subgroup* $H \leq G$*, we define the subgroups* $H_\alpha = H \cap G_\alpha$*. Show that* $H_{\alpha+1} \cong H_\alpha \oplus \mathbb{Z}$ *or* H_α *for every* α*.)*

HW1. *Prove that for a right* R*-module* M*, the Abelian group* $\mathrm{Hom}(R_R, M)$ *is also a right* R*-module with the action of* φr *(*$\varphi \in \mathrm{Hom}(R_R, M)$ *and* $r \in R$*):* $x(\varphi r) := (rx)\varphi$*.*

HW2. *Determine the number of projective modules with at most* 100 *elements over the ring* $\mathbb{Z}_{180}$ *up to isomorphism. Give another ring* R *for which the given Abelian groups are also projective as* R*-modules but there are other* R*-projectives with at most* 100 *elements.*