1. *Prove that the following hold in every category:*
a) if $\alpha\beta$ is an epimorphism then β is an epimorphism;
b) if $\alpha\beta$ is a monomorphis then α is a monomorphism.

Solution:
a) $\beta\gamma = \beta\delta \Rightarrow (\alpha\beta)\gamma = \alpha(\beta\gamma) = \alpha(\beta\delta) = (\alpha\beta)\delta$, but $\alpha\beta$ is an epimorphism, so $\gamma = \delta$.
b) $\gamma\alpha = \delta\alpha \Rightarrow \gamma(\alpha\beta) = (\gamma\alpha)\beta = (\delta\alpha)\beta = \delta(\alpha\beta)$, but $\alpha\beta$ is a monomorphism, so $\gamma = \delta$.

2. *Prove that the natural embedding $\mathbb{Z} \rightarrowtail \mathbb{Q}$ is an epimorphism in the category of rings with identity, though it is not surjective.*

Solution: In the category of rings with identity, morphisms preserve the identity element. We show that for any ring R in this category there is at most one morphism $\mathbb{Q} \to R$. $n\varphi = (1 + \ldots + 1)\varphi = 1\varphi + \ldots + 1\varphi = 1 + \ldots + 1 = n \cdot 1$, $0\varphi = 0$ and $(-n)\varphi = -(n\varphi) = -n \cdot 1$, for $n \in \mathbb{Z}_+$. Furthermore, $\frac{1}{n}\varphi$ is an inverse of $n\varphi$, since $(\frac{1}{n}\varphi)(n\varphi) = 1\varphi = 1$, and $(n\varphi)(\frac{1}{n}\varphi) = 1\varphi = 1$. But the inverse in a ring (even in a semigroup) is unique if it exists (if r' and r'' are both inverses of r, then $r' = r'rr'' = r''$), so $\frac{1}{n}\varphi$ is the unique inverse of n in R. Finally, $\frac{m}{n}\varphi = (m\varphi)(\frac{1}{n}\varphi)$ is also uniquely defined for every $m, n \in \mathbb{Z}$, $n \neq 0$. Thus the embedding of $\mathbb{Z}$ into $\mathbb{Q}$ must be an epimorphism.

Actually, the statement is true in the category of all rings, as well. Here any morphism $\varphi : \mathbb{Q} \to R$ is either 0 or an embedding, since $\mathbb{Q}$ has no proper ideal.

Suppose that $\mathbb{Z} \overset{\iota}{\rightarrowtail} \mathbb{Q}$ is the natural embedding, $\alpha, \beta : \mathbb{Q} \to R$, and $\iota\alpha = \iota\beta$. Then $1\alpha = 1\beta =: e \in R$ acts as an identity element in the subring S of R generated by $\operatorname{Im}\alpha \cup \operatorname{Im}\beta$, since it acts as an identity element on the generator set. So either $e = 0$, in which case $\alpha = \beta = 0$, or we may consider α and β as identity-preserving morphisms from $\mathbb{Q}$ to S, and then $\alpha = \beta$ by the first case.

3. *Prove that the product in a category is unique up to isomorphism and that in* Mod-R*, the product is the direct product.*

M is a product:
$$\begin{tikzcd} M \arrow[r, "\pi_i"] & M_i \\ \forall U \arrow[u, dashed, "\exists!\varphi"] \arrow[ur, "\alpha_i"'] & \end{tikzcd}$$

N is a coproduct:
$$\begin{tikzcd} N \arrow[d, dashed, "\exists!\varphi"'] & M_i \arrow[l, "\iota_i"'] \arrow[dl, "\alpha_i"] \\ \forall U & \end{tikzcd}$$

Solution: Suppose that both M and M' are a product of the modules M_i $(i \in I)$. Then $\exists\beta : M' \to M$ and $\beta' : M' \to M$ making the first diagram below commutative (i.e. $\beta\pi_i = \pi_i'$ and $\beta'\pi_i' = \pi_i$ for every $i \in I$).

$$\begin{tikzcd} M \arrow[r, "\pi_i"] \arrow[d, dashed, shift right, "\beta'"'] & M_i \\ M' \arrow[u, dashed, shift right, "\beta"'] \arrow[ur, "\pi_i'"'] & \end{tikzcd} \qquad \begin{tikzcd} M \arrow[r, "\pi_i"] \arrow[d, shift right, "1_M"'] \arrow[d, shift left, "\beta'\beta"] & M_i \\ M \arrow[ur, "\pi_i"'] & \end{tikzcd} \qquad \begin{tikzcd} M' \arrow[r, "\pi_i'"] & M_i \\ M' \arrow[u, shift left, "1_{M'}"] \arrow[u, shift right, "\beta\beta'"'] \arrow[ur, "\pi_i'"'] & \end{tikzcd}$$

But then $\beta'\beta\pi_i = \beta'\pi_i' = \pi_i$ and $\beta\beta'\pi_i' = \beta\pi_i = \pi_i'$, so we can make the second and the third diagram commutative both with $\beta'\beta$ and $\beta\beta'$, respectively, and the identity morphism. So by the uniqueness in the definition of the product, we get that $\beta'\beta = 1_M$ and $\beta\beta' = 1_{M'}$, i.e. $M \cong M'$.

Now, if we are in Mod-R, and M is the direct product of the modules M_i $(i \in I)$, then $u\varphi := (u\alpha_i)_{i\in I}$ is a morphism from U to M, clearly making the diagram in the definition commutative. On the other hand, if $\psi : U \to M$ makes the diagram commutative, and $u\psi = (v_i)_{i\in I}$, then $v_i = (u\psi)\pi_i = u(\psi\pi_i) = u\alpha_i$, so $u\psi = u\varphi$, consequently, $\psi = \varphi$.

4. *What are the epimorphisms and monomorphisms, injective and projective elements in the category of sets? Show that in this category the product and coproduct of two objects are usually not isomorphic.*

Solution: If $\varphi : A \to B$ is surjective, and $\alpha, \beta : B \to C$ such that $\varphi\alpha = \varphi\beta$, then for every $b \in B$ $\exists a \in A$ with $a\varphi = b$, so $b\alpha = a\varphi\alpha = a\varphi\beta = b\beta$, thus $\alpha = \beta$, showing that φ is an epimorphism.

If $\varphi : A \to B$ is not surjective, say, $\operatorname{Im}\varphi = U \subset B$, then let $b \in B \setminus U$ and $u \in U$, and define $\alpha : B \to B$ so that $x\alpha = x$ for every $x \neq b$ and $b\alpha = u$, and take $\beta = 1_B$. Then $\varphi\alpha = \varphi\beta$ but $\alpha \neq \beta$, so φ is not an epimorphism.

If $\varphi : A \to B$ is injective, and $\alpha, \beta : C \to A$ such that $\alpha\varphi = \beta\varphi$, then for every $a \in A$, $(a\alpha)\varphi = a(\alpha\varphi) = a(\beta\varphi) = (a\beta)\varphi \Rightarrow a\alpha = a\beta$, i.e. $\alpha = \beta$, showing that φ is a monomorphism.

If $\varphi : A \to B$ is not injective, that is, $a\varphi = a'\varphi$ for some $a \neq a' \in A$, then define $\alpha : A \to A$ so that $x\alpha = x$ for every $x \neq a'$ and $a'\alpha = a$, and take $\beta = 1_A$. Then $\alpha\varphi = \beta\varphi$ but $\alpha \neq \beta$, so φ is not a monomorphism.

We show that every set A is both projective and injective.

A is projective:
$$\begin{tikzcd} & A \arrow[dl, dashed, "\exists\gamma"'] \arrow[d, "\beta"] \\ X \arrow[r, two heads, "\alpha"] & Y \end{tikzcd}$$
A is injective:
$$\begin{tikzcd} X \arrow[r, tail, "\alpha"] \arrow[d, "\beta"'] & Y \arrow[dl, dashed, "\exists\gamma"] \\ A & \end{tikzcd}$$

In the first diagram let $a\gamma$ be an arbitrary inverse image of $a\beta$ by α, in the second, γ should map every $x\alpha \in Y$ to $x\beta$ and every $y \in Y \setminus \operatorname{Im}\alpha$ to any element of A.

By HW2, the product of sets is the Cartesian product, and the coproduct is the disjoint union. Two sets are isomorphic if and only if they have the same cardinality. So for a 2-element set and a 3-element set, the product has 6 elements, the coproduct 5, so they are not isomorphic.

5. *What is the coproduct of $\mathbb{Z}$ with itself, in the category of abelian groups and in the category of groups? What is the coproduct of $\mathbb{Z}_2$ with itself in the category of groups?*

Solution:
$$\begin{tikzcd} C_\infty \arrow[r, "\iota_1"] \arrow[dr, "\alpha_1"'] & F(x,y) \arrow[d, dashed, "\gamma"'] & C_\infty \arrow[l, "\iota_2"'] \arrow[dl, "\alpha_2"] \\ & G & \end{tikzcd}$$

Since the category of abelian groups is a module category, the coproduct here is the direct sum, i.e. $\mathbb{Z} \oplus \mathbb{Z}$. In the category of groups we use the multiplicative notation C_∞ instead of

$\mathbb{Z}$. Here the coproduct of C_∞ with itself is the free group $F(x, y)$ generated by two elements. If $C_\infty = \langle a \rangle$, let $\iota_1 : a \mapsto x$ and $\iota_2 : a \mapsto y$ (this can be extended to a homomorphism, since C_∞ itself is also a free group). Then for any group G and morphisms $\alpha_1, \alpha_2 : C_\infty \to G$, if $a\alpha_1 = g_1$ and $a\alpha_2 = g_2$, then there is a homomorphism $\gamma : F(x, y) \to G$ such that $x\gamma = g_1$ and $y\gamma = g_2$. So $a\iota_1\gamma = x\gamma = g_1 = a\alpha_1$ and $a\iota_2\gamma = y\gamma = g_2 = a\alpha_2$. Since a generates C_∞, this implies that $\alpha_i = \iota_i\gamma$ for $i = 1, 2$. So this γ makes the diagram commutative, and γ is unique, since the image of x and y is determined by the commutativity.

For the coproduct of C_2 with itself (we use again the multiplicative notation in the category of groups), we can take an appropriate factor group of the free group $F(x, y)$: factor out with the normal subgroup generated by x^2 and y^2, in other words, $K = \langle x, y \,|\, x^2 = y^2 = 1 \rangle$ will be the coproduct. Indeed, if there are two morphisms $\alpha_1, \alpha_2 : C_2 \to G$, or equivalently, if we pick two elements $g_1, g_2 \in G$ such that $g_1^2 = g_2^2 = 1$, then there is a homomorphism from K to G, mapping x to g_1 and y to g_2, since there is such a map mapping x to g_1 and y to g_2 from $F(x, y)$ to G, and its kernel must include the normal subgroup generated by x^2 and y^2. This map is unique because $K = \langle x, y \rangle$.

6. *Prove that the direct product of injective modules is injective and that any direct summand of an injective module is also injective. Which of these statements can be generalized to any category?*

Solution:

$$
\begin{tikzcd}
M \arrow[r, tail, "\alpha"] \arrow[d, "\beta"'] & N \arrow[dl, dashed, "\gamma"'] \arrow[ddl, dashed, "\gamma_i"] \\
\prod Q_i \arrow[d, "\pi_i"'] & \\
Q_i &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
M \arrow[r, tail, "\alpha"] \arrow[d, "\beta"'] & N \arrow[dl, dashed, "\gamma"'] \arrow[ddl, dashed, "\delta"] \\
U \arrow[d, shift right, "\iota_1"'] & \\
U \oplus V \arrow[u, shift right, "\pi_1"'] &
\end{tikzcd}
$$

Let Q_i $(i \in I)$ be injective modules, $\alpha : M \to N$ a monomorphism, and $\beta : M \to \prod Q_i$. By the injectivity of the Q_i's there is a morphism $\gamma_i : N \to Q_i$ for every i such that $\alpha\gamma_i = \beta\pi_i$. The definition of the product in categories states that there is a morphism $\gamma : N \to \prod Q_i$ with $\gamma\pi_i = \gamma_i$ for every i. Consequently, $\alpha\gamma\pi_i = \alpha\gamma_i = \beta\pi_i$ for each i. Now the uniqueness in the definition of the product ensures that $\alpha\gamma = \beta$. This proof did not use the special properties of module categories, only the definition of injective objects and products in categories.

Suppose that $U \oplus V$ is injective, $\alpha : M \to N$ is a monomorphism, $\beta : M \to U$, and $U \xrightarrow{\iota_1} U \oplus V \xrightarrow{\pi_1} U$ the embedding and projection for the first component of the direct sum. By the injectivity of $U \oplus V$, there exists a $\delta : N \to U \oplus V$ such that $\alpha\delta = \beta\iota_1$. But then for $\gamma := \delta\pi_1$, we have $\alpha\gamma = \alpha\delta\pi_1 = \beta\iota_1\pi_1 = \beta$. Here we used that $U \oplus V$ is both a coproduct and a product, which is not true in categories in general, but it is true in a module category.

7. *Show that every vector space is projective and injective*

Solution: The proof is practically the same as for sets, we use that every vector space is free.

A is projective:
$$\begin{tikzcd} & V \arrow[dl, dashed, "\exists\gamma"'] \arrow[d, "\beta"] \\ X \arrow[r, two heads, "\alpha"] & Y \end{tikzcd}$$
A is injective:
$$\begin{tikzcd} X \arrow[r, tail, "\alpha"] \arrow[d, "\beta"'] & Y \arrow[dl, dashed, "\exists\gamma"] \\ V & \end{tikzcd}$$

In the first diagram, we define γ on a basis of V such that $b_i\gamma$ is an arbitrary inverse image of $b_i\beta$ by α, and then we extend γ to the whole vector space. Since $\beta = \gamma\alpha$ on the basis elements, and α, β, γ are linear, $\beta = \gamma\alpha$ on the whole V.

In the second diagram we take a direct complement to $\operatorname{Im}\alpha$ in Y: $Y = \operatorname{Im}\alpha \oplus U$, and define γ on $\operatorname{Im}\alpha$ as $\alpha^{-1}\beta$ and on U to be the zero map.

8. *Suppose that G is a divisible abelian group.*
  a) *Prove that for any element $g \in G$, if $o(g) = \infty$, then g is included in a direct summand of G isomorphic to $\mathbb{Q}$, if $o(g) = p^n$ for some p prime and $n \geq 1$, then g is included in a direct summand of G isomorphic to $\mathbb{Z}_{p^\infty}$.*
  b) *Prove that G is the direct some of some copies of $\mathbb{Q}$ $\mathbb{Z}_{p^\infty}$ (p any prime).*

Solution: a) Let $I = \mathbb{Q}$ or $\mathbb{Z}_\infty$ in the two cases. Embed $H = \langle g \rangle$ into I with α, and into G with the natural embedding, ι. Then by the injectivity of G, there is a morphism $\gamma : G \to I$ such that $\iota = \alpha\gamma$ and $\operatorname{Im}\alpha$ is an essential submodule of I, i.e. every nonzero submodule intersects $\operatorname{Im}\alpha$. Now $0 = \operatorname{Ker}\iota = (\operatorname{Ker}\gamma \cap \operatorname{Im}\alpha)\alpha^{-1}$ implies $\operatorname{Ker}\gamma = 0$, so γ is injective. But I is an injective module, so $\operatorname{Im}\gamma \cong I$ is a direct summand of G.

b) Let G be an injective abelian group, and let $\{\, G_i \,|\, i \in I \,\}$ be the set of all subgroups of G which are isomorphic to either $\mathbb{Q}$ or some of the $\mathbb{Z}_{p^\infty}$. By Zorn's lemma there is a maximal subset I_0 of I belonging to independent subgroups, i.e. such that $H := \sum\limits_{i \in I_0} G_i = \bigoplus\limits_{i \in I} G_i$. Since the sum of divisible groups is also divisible, H is a divisible group, so it is injective. But then H is a direct summand of G and its direct complement K is also injective. If $K \neq 0$, then by part a) it contains a subgroup isomorphic to $\mathbb{Q}$ or some $\mathbb{Z}_{p^\infty}$, contradicting to the maximality of I_0.

HW1. *Prove that among the $\mathbb{Z}_4$-modules, $\mathbb{Z}_4$ is injective but $\mathbb{Z}_2$ is not.*

HW2. *Prove that in the category of sets the product is the cartesian product, and the coproduct is the disjoint union of sets.*