1. *Suppose $e \in R$ is an idempotent element. Prove that $R_R = eR \oplus (1-e)R$.*

Solution: It is enough to show that $\{ e, 1-e \}$ is a complete set of orthogonal idempotents. Indeed, e is idempotent by assumption, $(1-e)^2 = 1-2e+e^2 = 1-2e+e = 1-e$, so $1-e$ is also idempotent, $e(1-e) = (1-e)e = e-e^2 = 0$, so they are orthogonal, finally, $e+(1-e) = 1$, so they form a complete set of idempotents.

2. *Write R_R as a direct sum of indecomposable modules. What are the dimensions of the direct components?*

a) $R = KC_4$, $K = \mathbb{Z}_2$
b) $R = KC_4$, $K = \mathbb{R}$
c) $R = KC_4$, $K = \mathbb{C}$
d) R is the ring of 3×3 upper triangular matrices over $\mathbb{R}$

Solution: In a), b) and c), let the group $G \cong C_4$ be generated by the element a. Then $KG = \{ x + ya + za^2 + ua^3 \mid x, y, z, u \in K \}$. For the decomposition, we need to find idempotents different from 0 and 1.

a) Here we can use that $x^2 = x$ and $2x = 0$ for every $x \in \mathbb{Z}_2$. So $(x+ya+za^2+ua^3)^2 = x + ya^2 + z + ua^2 = (x+z) + (y+u)a^2$, and this is equal to $x + ya + za^2 + ua^3$ if and only if $y = z = u = 0$, so this element is either 0 or 1. This means that KG_{KG} is indecomposable.

b) Note that for any $H \leq G$, the sum of the elements s_H of H is nilpotent or almost idempotent: $s_H h' = (\sum_{h \in H} h)h' = \sum_{h \in H} hh' = s_H$, so $s_H^2 = |H| s_H$. If $\operatorname{char} K \mid |H|$, then this gives $s_H^2 = 0$, if not, then $e = \frac{1}{|H|} s_H$ is idempotent.
Use this for $H = \{ 1, a^2 \}$. Then we get that $e = \frac{1}{2}(1+a^2)$ and $1-e = \frac{1}{2}(1-a^2)$ form a complete set of orthogonal idempotents. But $f = \frac{1}{4} s_G$ is also idempotent, and it is in eKG (since $1 + a + a^2 + a^3 = (1+a^2)(1+a)$). But $eKG = e^2KG = eKGe$, so by the statement of HW1, f, $e-f$ and $1-e$ generate direct components of the regular module. The dimensions of the three summands are 1, 1 and 2, so the only question is whether $(1-e)KG$ is decomposable. If we try to find a decomposition of the last summand, we have to solve the equation $((1-a^2)(x+ya))^2 = (1-a^2)(x+ya)$, that is, $2x^2 - 2y^2 = x$ and $4xy = y$, which gives either $y = 0$ and $x = 0$ or $\frac{1}{2}$, producing the trivial idempotents, or it leads to an equation $2y^2 = -\frac{1}{8}$, which has no solution in $\mathbb{R}$. So the three components are indecomposable.

c) The system of idempotents found in part b) also give a decomposition of $\mathbb{C}G$, but here the last idempotent is decomposable: the equation in the previous part gives $x = \frac{1}{4}$ and $y = \frac{1}{4}i$, and this produces two more 1-dimensional components: $(1-a^2)KG = (1+ia)(1-a^2)KG \oplus (1-ia)(1-a^2)KG$. The four components are 1-dimensional, so they must be simple modules, thus indecomposable.

d) The idempotent matrices $e_1 = diag(1,0,0)$, $e_2 = diag(0,1,0)$ and $e_3 = diag(0,0,1)$ form a complete set of orthogonal idempotents, and HW1 shows that the components are indecomposable, since $e_i R e_i = e_i \mathbb{R}$ is one-dimensional, and its only idempotents are 0 and e_i. The component $e_i R$ consists of those matrices of R which have all 0's outside its i'th row, so the dimensions of the components are 3, 2 and 1.

3. *Let $0 = M_0 < M_1 < \ldots < M_{k-1} < M_k = M$ be a composition series of the module M,* *and let U be a submodule of M. Prove that the factors of the series*

$$0 = M_0 \cap U \leq M_1 \cap U \leq \ldots \leq M_{k-1} \cap U \leq M_k \cap U = U \text{ and}$$

$$0 = (M_0 + U)/U \leq (M_1 + U)/U \leq \ldots \leq (M_{k-1} + U)/U \leq (M_k + U)/U = M/U$$

are all simple or zero modules, and at each step the factor is zero in exactly one of the two series, and isomorphic to the corresponding factor M_i/M_{i-1} in the other.

Solution:

$$\begin{aligned}(M_{i+1} \cap U)/(M_i \cap U) =&(M_{i+1} \cap U)/((M_{i+1} \cap M_i) \cap U) = (M_{i+1} \cap U)/(M_i \cap (M_{i+1} \cap U)) \\ \cong&((M_{i+1} \cap U) + M_i)/M_i \cong (M_{i+1} \cap (U + M_i))/M_i \leq M_{i+1}/M_i\end{aligned}$$

(using the first isomorphism theorem and the modular identity from Problem Set 1/5). The last module is supposed to be simple, so $(M_{i+1} \cap U)/(M_i \cap U)$ is either zero or $\cong M_{i+1}/M_i$.

$$\begin{aligned}((M_{i+1} + U)/U)/((M_i + U)/U) \cong&(M_{i+1} + U)/(M_i + U) = ((M_i + M_{i+1}) + U)/(M_i + U) \\ =&((M_i + U) + M_{i+1})/(M_i + U) \\ \cong&M_{i+1}/((M_i + U) \cap M_{i+1}) \\ =&M_{i+1}/(M_i + (U \cap M_{i+1})),\end{aligned}$$

and this is a factor of M_{i+1}/M_i by the second isomorphism theorem, so $((M_{i+1} + U)/U)/((M_i + U)/U)$ is either zero or it is isomorphic to M_{i+1}/M_i.

If the ith factor of the first series is zero, then $(M_{i+1} \cap U) + M_i = M_i$, so $M_{i+1} \cap U \leq M_i$, so the factor of the second series is isomorphic to $M_{i+1}/(M_i + (U \cap M_{i+1}) = M_{i+1}/M_i$.

If the factor of the second series is zero, then $(M_i + U) \cap M_{i+1} = M_{i+1}$, so $M_{i+1} \leq M_i + U$, so the factor of the first series is isomorphic to $(M_{i+1} \cap (U + M_i)/M_i = M_{i+1}/M_i$.

4. *Prove that*

a) $\mathrm{Hom}(\bigoplus_{i\in I} M_i, N) \cong \prod_{i\in I} \mathrm{Hom}(M_i, N)$*, and*

b) $\mathrm{Hom}(M, \prod_{i\in I} N_i) \cong \prod_{i\in I} \mathrm{Hom}(M, N_i)$.

Solution: a) Let $M_i \xrightarrow{\iota_i} \bigoplus_{i\in I} M_i$ be the embedding of the i'th component. Then to any morphism $\varphi : \bigoplus_{i\in I} M_i \to N$, we can assign $(\iota_i\varphi)_{i\in I} \in \prod_{i\in I} \mathrm{Hom}(M_i, N)$, which is clearly a homomorphism of abelian groups. This homomorphism is also bijective, because for any $(\psi_i)_{i\in I} \in \prod_{i\in I} \mathrm{Hom}(M_i, N)$, there is a unique morphism $\varphi : \bigoplus_{i\in I} M_i \to N$ such that $\psi_i = \iota_i\varphi$ for every i, according to the categorical definition of a coproduct.

b) Let $\prod_{i\in I} N_i \xrightarrow{\pi_i} N_i$ be the projection to the i'th component. Then to any $\varphi : M \to \prod_{i\in I} N_i$, we can assign $(\varphi\pi_i)_{i\in I} \in \prod_{i\in I} \mathrm{Hom}(M, N_i)$, which is clearly a homomorphism of abelian groups. This homomorphism is also bijective, because for any $(\psi_i)_{i\in I} \in \prod_{i\in I} \mathrm{Hom}(M, N_i)$, there is a unique morphism $\varphi : M \to \prod_{i\in I} N_i$ such that $\psi_i = \varphi\pi_i$ for every i, according to the categorical definition of a product.

5. *Suppose that for a submodule $U \leq R_R$, the factor module R_R/U is semisimple. Prove that $U \geq J(R)$. Give an example when $R_R/J(R)$ is not semisimple.*

Solution: If $R_R/U = \bigoplus_{i\in I} S_i$, where S_i are simple, then for every i there is an epimorphism $\tilde{\varphi}_i : R_R/U \to S_i$ such that $\bigcap_{i\in I} \operatorname{Ker} \tilde{\varphi}_i = \bar{0}$, and then for the natural extensions $\varphi_i : R_R \to S_i$ of $\tilde{\varphi}_i$, we have $\bigcap_{i\in I} \operatorname{Ker} \varphi_i \leq U$. Since $\operatorname{Im} \varphi_i$ are simple, this intersection contains $J(R)$, thus $J(R) \leq U$.

For $R = \mathbb{Z}$, the maximal ideals are $p\mathbb{Z}$, where p are primes, and $\bigcap_{p \text{ prime}} p\mathbb{Z} = 0$ (there is no nonzero integer which is divisible by all primes), so $J(\mathbb{Z}) = 0$. But $\mathbb{Z}/J(\mathbb{Z}) = \mathbb{Z}$ is not semisimple, since it cannot be written as a direct sum of simple modules (such a direct sum would have only elements of finite order).

A right or left ideal I of R is nilpotent if there is an integer $k > 0$ such that $I^k = 0$

6. *Prove that the following three statements are equivalent for a right ideal J of a finite dimensional algebra A.*
(i) J is nilpotent.
(ii) Every simple A-module is annihilated by J.
(iii) Every finite dimensional A-module is annihilated by an appropriate power of J.

Solution: (i)⇒(ii): Suppose S is simple. Then $SJ \leq S \Rightarrow SJ = 0$ or $SJ = S$. But if $SJ = S$, then $S = SJ = SJJ = \ldots = SJ^k = S0 = 0$ for a large enough k, which is a contradiction.
(ii)⇒(iii): A finite dimensional module has a finite composition series $0 = M_0 < M_1 < \ldots < M_k = M$, and by condition (ii), J annililates every composition factor, i.e. $M_{i+1}J \leq M_i$ for every i, so $MJ^k \leq M_{k-1}J^{k-1} \leq \ldots \leq M_1 J = 0$.
(iii)⇒(i): We can apply condition (iii) to A_A: there is a k such that $0 = AJ^k \supseteq 1J^k = J^k$.

7. *Prove the following statements for the Jacobson radical of a finite dimensional algebra A.*
a) $J(A)$ annihilates all (semi)simple modules.
b) $J(A)$ is the smallest right ideal such that $A_A/J(A)$ is semisimple (i.e. it is contained by all the other right ideals with this property).
c) $J(A)$ is the largest nilpotent right ideal (i.e. it contains all the other nilpotent right ideals).
d) $J(A)$ is the only right ideal with the property that $A_A/J(A)$ is semisimple and $J(A)$ is nilpotent.
e) $J(A)$ is a two-sided ideal.

Solution: a) Let M be a simple module. Then any $0 \neq m \in M$ generates M, i.e. $mA = M$. There is a homomorphism $\varphi : A_A \to M$ such that $1 \mapsto m$. Its kernel is maximal in A_A, so it contains $J(A)$. But then $0 = J(A)\varphi = 1\varphi J(A) = mJ(A)$, thus $J(A)$ annihilates every element of M. It follows from this that $(\oplus S_i)J = \oplus(S_i J) = \oplus 0 = 0$ if all modules S_i are simple.

b) $J(A)$ is the intersection of all maximal right ideals, but since A is finite dimensional, it is the intersection of finitely many maximal right ideals. Thus $A/J(A)$ can be embedded into the direct product of finitely many simple modules, but that is also

the direct sum of those simple modules, so $A/J(A)$ can be embedded into a semisimple module, which implies that $A/J(A)$ itself is semisimple.
On the other hand, if the factor by some right ideal U is semisimple then by Problem 5, $J(A) \leq U$.

c) From part a), and Problem 6 it follows that $J(A)$ is nilpotent. On the other hand, every nilpotent right ideal U annihilates all simple modules, and so all semisimple modules, as well, thus $(A/J(A))U = \bar{0}$, and this means that $U \leq AU \leq J(A)$.

d) Part b) and c) shows that $J(A)$ is nilpotent, and $A/J(A)$ is semisimple. On the other hand, if some $U \leq A_A$ is nilpotent, and A/U is a semisimple module, then by part b) and c), $J(A) \leq U$ and $U \leq J(A)$, so $U = J(A)$.

e) $J(A)^k = 0$ implies that $(AJ(A))^k = A(J(A)A)^{k-1}J(A) \leq AJ(A)^k = A0 = 0$, so $AJ(A)$ is also nilpotent. But $J(A)$ contains all nilpotent right ideals by part c), so $AJ(A) \leq J(A)$.

8. *Which of the rings in problem 2 are semisimple? What is the Jacobson radical in each case?*

Solution: We use the notation A instead of R, to remind ourselves that all four rings are actually finite dimensional algebras.

a) Let us observe that $(1-a)^4 = 0$, and A is commutative, so $((1-a)A)^4 = (1-a)^4A^4 \leq 0A = 0$, thus $(1-a)A \leq J(A)$ by 7.c). This ideal also contains $1-a^2 = (1-a)(1+a)$ and $1-a^3 = (1-a)(1+a+a^2)$, so it is at least 3-dimensional. But $J(A) \neq A$ if A is a finite dimensional algebra, so $J(A) = (1-a)A$. In fact, this ideal consists of those elements, for which the sum of the coefficients is 0. This also means that A is not semisimple.

b) $J(A)$ is the direct sum of the Jacobson radical of the three components, so it is either 1-dimensional or 0, depending on whether the 2-dimensional component is simple or not. But there is a 2-dimensional simple module over $\mathbb{R}C_4$: the generator element a can act on $\mathbb{R}^2$ by the matrix $m = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. The order of this matrix is 4, so $a \mapsto m$ gives a homomorphism from $\mathbb{R}C_4$ to $\operatorname{End}\mathbb{R}^2$, and this vector space has no nontrivial m-invariant subspace, since m has no real eigenvector. So A_A must have a 2-dimensional simple module among its composition factors, thus the last component must be simple, so A is semisimple and $J(A) = 0$.

c) The solution of 2.c) shows that A_A is the direct sum of four 1-dimensional simple modules, so A is semisimple and $J(A) = 0$.

d) The upper triangular matrices whose diagonal elements are all 0 clearly form an ideal I, and this ideal is also nilpotent (its third power is 0), so this ideal is included in $J(A)$. On the other hand, $A/I \cong \{\text{ diagonal matrices }\} \cong \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R}$ is semisimple, so $J(A) = I$.

HW1. *Suppose that M is a direct summand of R_R, i.e. $M = eR$ for some idempotent element e. Prove that M is decomposable if and only if there is an idempotent element $f \in eRe$ such that $0 \neq f \neq e$.*

HW2. *Prove that* $\operatorname{Hom}(M, \bigoplus_{i\in I} N_i) \cong \bigoplus_{i\in I} \operatorname{Hom}(M, N_i)$ *if M is finitely generated.*