1. *a) Prove that for any ring R, if $U, V \leq R_R$ such that R_R/U is semisimple and V is nilpotent then $V \leq J(R) \leq U$.*
b) Prove that $\mathbb{Z}/J(\mathbb{Z})$ is not semisimple.
c) Let $R = \left\{ \frac{a}{b} \mid a, b \in \mathbb{Z}, b \text{ is odd} \right\}$ and $I = 2R$. Prove that $I = J(R)$, and I is not nilpotent.

Solution: a) If $R_R/U = \bigoplus_{i\in I} S_i$, where S_i are simple, then for every i there is an epimorphism $\tilde{\varphi}_i : R_R/U \to S_i$ such that $\bigcap_{i\in I} \operatorname{Ker} \tilde{\varphi}_i = \bar{0}$, and then for the natural extensions $\varphi_i : R_R \to S_i$ of $\tilde{\varphi}_i$, we have $\bigcap_{i\in I} \operatorname{Ker} \varphi_i \leq U$. Since $\operatorname{Im} \varphi_i$ are simple, this intersection contains $J(R)$, thus $J(R) \leq U$.
If V is nilpotent, say, $V^k = 0$, then V annihilates every simple module: if S is simple and $SV \neq 0$, then $SV = S$, so $S = SV = SVV = \ldots = SV^k = 0$, a contradiction. Thus $V \leq RV \leq M$ for every maximal submodule M of R_R, which gives that $V \leq J(R)$.

b) For $R = \mathbb{Z}$, the maximal ideals are $p\mathbb{Z}$, where p are primes, and $\bigcap_{p \text{ prime}} p\mathbb{Z} = 0$ (there is no nonzero integer which is divisible by all primes), so $J(\mathbb{Z}) = 0$. But $\mathbb{Z}/J(\mathbb{Z}) = \mathbb{Z}$ is not semisimple, since it cannot be written as a direct sum of simple modules, i.e. of cyclic groups of prime order (such a direct sum would have only elements of finite order).

c) I is an ideal of R_R, and every element of $R \setminus I$ is invertible, so every proper (right) ideal is included in I. This means that I is the unique maximal right ideal of R, so $I = J(R)$. On the other hand, R has no nilpotent elements, so it cannot have a nontrivial nilpotent (right) ideal.

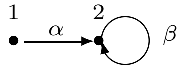
2. *Prove that the following statements are equivalent for a ring R.*
a) R is semisimple.
b) Every R-module is projective.
c) Every R-module is injective.

Solution: We know that the following statements are equivalent to a), b) and c), respectively.
a') In every R-module every submodule is a direct summand.
b') For every epimorphism $\varphi : M \to N$, $\operatorname{Ker} \varphi$ is a direct summand of M.
c') For every monomorphism $\varphi : M \to N$, $\operatorname{Im} \varphi$ is a direct summand of N.
It is clear that a') implies both b') and c'). On the other hand, for any $U \leq M$, U is the kernel of the natural epimorphism $M \to M/U$, and the image of the natural embedding $U \to M$, so if either b') or c') holds, then a') also holds.

3. *Let $e_1, \ldots, e_n$ be a complete set of orthogonal idempotents in A, and $M \in$ Mod-A. Take the decomposition $M = M_1 \oplus \ldots M_n$ of M into a direct sum of subspaces $M_i = Me_i$. Show that the elements of e_iAe_j act as linear maps from M_i to M_j, and the action of e_iAe_j $(i, j = 1, \ldots, n)$ determines the action of A on M.*

Solution: M is indeed the direct sum of the vector spaces M_i, since every element $m \in M$ can be written as $m1 = me_1 + me_2 + \ldots + me_n$, and for any element $me_i \in M_i$, $(me_i)e_i = me_i$ but $M_je_i = Me_je_i = 0$, so $M_i \cap \sum_{j\neq i} M_j = 0$.

For an element $a \in e_i A e_j$, $M_k a = M e_k a = 0$ if $k \neq i$, and $Ma \leq M_j$, so the action of a is determined by its restriction to M_i, and it maps to M_j.
A can be written as the direct sum of the subspaces $e_i A e_j$, so the action $m \mapsto ma$, where $a = \sum_{i,j} e_i a e_j$ is determined by the action of the components $e_i a e_j$, mapping M_i to M_j.

4. *Consider the graph algebra $K\Gamma/I$, where Γ :* 

and $I = (\alpha\beta^2, \beta^3)$. Let $M = M_1 \oplus M_2$ be a vector space such that $\dim_K M_1 = \dim_K M_2 = 2$, and fix a basis $\mathcal{B} = \{ b_1, b_2 \}$ in M_1 and $\mathcal{C} = \{ c_1, c_2 \}$ in M_2. We define the action of A as $xe_i = x$ if $x \in M_i$ and 0 if $x \in M_j$ $(j \neq i)$, the matrix of $M_1 \xrightarrow{\alpha} M_2$ in $(\mathcal{B}, \mathcal{C})$ is $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, and the matrix of $M_2 \xrightarrow{\beta} M_2$ in $(\mathcal{C}, \mathcal{C})$ is $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

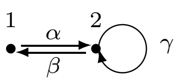
a) Show that with the natural extension of the action of A, M becomes an A-module.
b) Determine the Loewy diagram of M, using the basis $\mathcal{B} \cup \mathcal{C}$.
c) Find the Loewy diagram of the submodule U of M generated by $b_1 - b_2$, and the Loewy diagram of the factor module M/U.

Solution: a) If we define the action of the arrows (and the idempotents) so that an arrow $i \mapsto j$ is a linear map from M_i to M_j and the ith idempotent acts as the identity map on M_i, while they are 0 on all the other components, and then extend this action naturally to the paths and linear combinations of paths, then we clearly get a $K\Gamma$-module. So we only have to check that the generator elements (and consequently all elements) of the ideal I annihilate M. Indeed, the matrix of $\beta^2 : M_2 \to M_2$ is $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}^2 = 0$, so both $\alpha\beta^2$ and β^3 acts as the 0 map. This means by Problem 1/3.b) that M is also a $K\Gamma/I$-module.

b) We can see from the given matrices that $b_1 \underset{\alpha}{\longrightarrow} c_2$, $b_2 \underset{\alpha}{\longrightarrow} c_1$, $c_1 \underset{\beta}{\longrightarrow} c_2$, and $c_2 \underset{\beta}{\longrightarrow} 0$ (we write matrices on the right, so the images of the basis elements are coded in the rows of the matrix). In a diagram:

$$\begin{array}{ccc} & b_2 & \\ & \downarrow\alpha & \\ b_1 & c_1 & \\ \alpha\searrow & \swarrow\beta & \\ & c_2 & \end{array} \quad \text{simpler:} \quad \begin{array}{ccc} & 1 & \\ & \downarrow\alpha & \\ 1 & 2 & \\ \alpha\searrow & \swarrow\beta & \\ & 2 & \end{array} \quad \text{even simpler:} \quad \begin{array}{ccc} & 1 & \\ & | & \\ 1 & 2 & \\ \diagdown & \diagup & \\ & 2 & \end{array} \quad \text{or} \quad \begin{smallmatrix} & 1 \\ 1 & 2 \\ 2 & \end{smallmatrix}$$

c) $b_1 - b_2 \in M_1$, $(b_1 - b_2)\alpha = c_2 - c_1 \in M_2$, $(c_2 - c_1)\beta = -c_2 \in M_2$, $-c_2\beta = 0$, and $b_1 - b_2$, $c_2 - c_1$ and $-c_2$ are linearly independent, so the Loewy diagram of U is $\begin{smallmatrix} 1 \\ 2 \\ 2 \end{smallmatrix}$. On the other hand, M/U is generated by $b_1 + U \in \bar{M}_1$ such that $b_1\alpha \in U$ (and naturally, $b_1\beta = 0$), so M/U is the simple module with Loewy diagram 1. (Actually, the basis $\{ -b_1,\ b_1 - b_2,\ c_2 - c_1,\ -c_2 \}$ gives the same Loewy diagram for M as the original, and then we can see the submodule and the factor as part of the diagram for M.)

5. *Give a basis and the Loewy diagram of the indecomposable direct summands of A_A if $A = K\Gamma/I$, where Γ:* 

and

a) $I = (\alpha\gamma, \gamma^2, \gamma\beta, \alpha\beta\alpha, \beta\alpha\beta)$; *b) $I = (\alpha\gamma^2, \gamma^2 - \beta\alpha, \alpha\beta)$.*

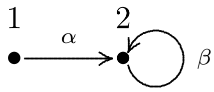
Solution: a) $A_A = \begin{smallmatrix} 1 \\ 2 \\ 1 \end{smallmatrix} \oplus \begin{smallmatrix} & 2 \\ 1 & & 2 \\ & 2 \end{smallmatrix}$

b) $A_A = \begin{smallmatrix} 1 \\ 2 \\ 2 \\ 1 \end{smallmatrix} \oplus \begin{smallmatrix} & 2 & \\ 1 & & 2 \\ & 2 & & 1 \\ & & 2 \end{smallmatrix}$.

6. *Is there a graph algebra $K\Gamma/I$ such that the Loewy diagram of the regular module is the one shown below? If yes, give the graph Γ and a generator system of the ideal I.*

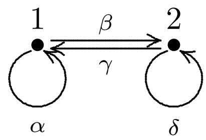
a) $\begin{smallmatrix} 1 \\ 2 \\ 2 \\ 2 \end{smallmatrix} \oplus \begin{smallmatrix} 2 \\ 2 \end{smallmatrix}$ *b)* $\begin{smallmatrix} & 1 & \\ 1 & & 2 \\ & 2 & \end{smallmatrix} \oplus \begin{smallmatrix} & 2 & \\ 1 & & 2 \end{smallmatrix}$ *c)* $\begin{smallmatrix} 1 \\ 2 \end{smallmatrix} \oplus \begin{smallmatrix} 2 \\ 3 \end{smallmatrix} \oplus 3$

Solution: a) There is no such algebra, since the graph of that algebra could only be



but it can be seen from the second component that $\beta^2 = 0$, while in the first component $\alpha\beta^2 \neq 0$, giving a contradiction.

b) The graph of the algebra is

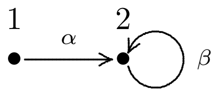


and $I = (\alpha^2, \beta\gamma, \alpha\beta - \beta\delta, \gamma\alpha, \gamma\beta, \delta^2, \delta\gamma)$.

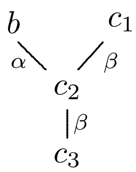
c) The graph of the algebra is $1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3$, and $I = (\alpha\beta)$.

7. *Let $A = K\Gamma/I$ be a graph algebra with the Loewy diagram $A_A = \begin{smallmatrix} 1 \\ 2 \\ 2 \end{smallmatrix} \oplus \begin{smallmatrix} 2 \\ 2 \\ 2 \end{smallmatrix}$ from problem 4. Determine all the submodules and their factors for the module $\begin{smallmatrix} 1 & & 2 \\ & 2 & \\ & 2 & \end{smallmatrix}$.*

Solution: The graph of the algebra is Γ :



and $A = K\Gamma/(\alpha\beta^2, \beta^3)$. Let the basis elements of $\begin{smallmatrix} 1 & & 2 \\ & 2 & \\ & 2 & \end{smallmatrix}$ be b, c_1, c_2, c_3, where $M_1 := Me_1 = \langle b \rangle_K$, $M_2 := Me_2 = \langle c_1, c_2, c_3 \rangle_K$ such that the module is



Let $U \leq M$, $U_1 := Ue_1 \leq M_1$ and $U_2 := Ue_2 \leq M_2$. If $U_1 \neq 0$, then $U_1 = M_1 = \langle b \rangle_K$, so $bA = \langle b, c_2, c_3 \rangle_K \leq U$, thus either $U = M$ or U is 3-dimensional with Loewy diagram $\begin{smallmatrix} 1 \\ 2 \\ 2 \end{smallmatrix}$. Now suppose that $U_1 = 0$, so $U \leq M_2 = c_1 A$. If U contains an element $u = \lambda c_1 + \mu c_2 + \nu c_3$, where $\lambda \neq 0$, then $u\beta = \lambda c_2 + \mu c_3 \in U$ and $u\beta^2 = \lambda c_3$, and these generate the whole

$M_2 = \begin{smallmatrix}2\\2\\2\end{smallmatrix}$. If $\lambda = 0$ for all $u \in U$, but $\mu \neq 0$ for some, then $u\beta = \mu c_3 \in U$ gives that $U = \langle c_2, c_3 \rangle_K = \begin{smallmatrix}2\\2\end{smallmatrix}$. If $\lambda = \mu = 0$ for all $u \in U$ but $U \neq 0$, then $U = \langle c_3 \rangle = 1$. Finally, U may be the zero module. So M has only 6 submodules, with Loewy diagrams:

$$\begin{smallmatrix}1&&2\\&2&\\&2&\end{smallmatrix}, \quad \begin{smallmatrix}1\\2\\2\end{smallmatrix}, \quad \begin{smallmatrix}2\\2\\2\end{smallmatrix}, \quad \begin{smallmatrix}2\\2\end{smallmatrix}, \quad 2, \quad 0,$$

and their factor modules are

$$0, \quad 2, \quad 1, \quad 1 \oplus 2, \quad \begin{smallmatrix}1&&2\\&2&\end{smallmatrix}, \quad \begin{smallmatrix}1&&2\\&2&\\&2&\end{smallmatrix}.$$

8. *What is the dimension of the vector spaces* $\mathrm{Hom}(\begin{smallmatrix}&2&\\1&&2\end{smallmatrix}, \begin{smallmatrix}1&&2\\&2&\end{smallmatrix})$, *and* $\mathrm{Hom}(\begin{smallmatrix}1&&2\\&2&\end{smallmatrix}, \begin{smallmatrix}&2&\\1&&2\end{smallmatrix})$ *for modules over the algebra of problem 5.a).*

Solution: Let $M = \begin{smallmatrix}&2&\\1&&2\end{smallmatrix}$ and $N = \begin{smallmatrix}1&&2\\&2&\end{smallmatrix}$. The generator element of M is in Me_2, so it can only be mapped into the two-dimensional space Ne_2, and there exist two such independent homomorphisms: if we factor M with the simple submodule 1, then the factor module $\begin{smallmatrix}2\\2\end{smallmatrix}$ can be embedded into N, or we can embed $M/\operatorname{rad} M = 2$ into the socle of N. So $\dim_K \mathrm{Hom}(\begin{smallmatrix}&2&\\1&&2\end{smallmatrix}, \begin{smallmatrix}1&&2\\&2&\end{smallmatrix}) = 2$.

There is no monomorphism of N to M (since both modules are 3-dimensional, this would be an isomorphism), and every nonzero submodule of N contains the socle 2, so the homomorphisms from N to M go through $1 \oplus 2 \to \begin{smallmatrix}&2&\\1&&2\end{smallmatrix}$. But $1 \oplus 2$ can only go into the socle of M. So $\mathrm{Hom}(N, M) \cong \mathrm{End}(1 \oplus 2)$, and the latter is generated by the independent morphisms $\pi_1\iota_1$ and $\pi_2\iota_2$. So $\dim_K \mathrm{Hom}(\begin{smallmatrix}1&&2\\&2&\end{smallmatrix}, \begin{smallmatrix}&2&\\1&&2\end{smallmatrix}) = 2$.

Hf1. *Determine the Jacobson radical of the ring R of 3×3 upper triangular matrices over $\mathbb{Z}_2$, and the radical of the R-module M of all 3×3 matrices over $\mathbb{Z}_2$.*

Hf2. *What is the Loewy diagram of the regular module of $A = K\Gamma/I$ if*

$$\Gamma : 1 \underset{\beta}{\overset{\alpha}{\rightleftarrows}} 2 \underset{\delta}{\overset{\gamma}{\rightleftarrows}} 3, \qquad I = (\alpha\gamma, \beta\alpha - \gamma\delta, \delta\beta)?$$