1. *Find the graph algebra which is isomorphic to the algebra of 2×2 upper triangular matrices over K.*

Solution: This is a 3-dimensional algebra with basis $\{ E_{11}, E_{12}, E_{22} \}$, where E_{ij} denotes the matrix with a single 1 at the position (i, j) and 0 everywhere else. The products $E_{ij} E_{k\ell} = \delta_{jk} E_{i\ell}$ are the same as in the path algebra of two vertices and one arrow α from 1 to 2, if we use the map $E_{11} \mapsto e_1$, $E_{22} \mapsto e_2$ and $E_{12} \mapsto \alpha$. So the linear extension gives an isomorphism between these two algebras.

2. *Show that for $A = K^{n \times n}$, the regular module A_A is the direct sum of n isomorphic simple modules.*

Solution: $1 = E_{11} + E_{22} + \ldots + E_{nn}$ gives a direct decomposition of $K^{n \times n}$. The components $A_i := E_{ii} K^{n \times n}$ consist of the matrices with zeros outside of the ith row. Any matrix $0 \neq M \in A_i$ generates the whole component: if $m_{ij} \neq 0$ for some j then $E_{ik} = M(m_{ij}^{-1} E_{jk}) \in M K^{n \times n}$ for every k, so $M K^{n \times n} = A_i$. This proves that the n components are all simple. The multiplication by E_{ij} on the left is a module homomorphism from A_i to A_j, whose inverse is the multiplication by E_{ji} on the left. So $A_i \cong A_j$ for all i, j.

3. *Let $M \in \text{mod-}A$ be indecomposable. Use the Fitting Lemma to prove that $R = \operatorname{End}(M)$ is a local ring, i.e. $R/J(R)$ is a division ring.*

Solution: By the Fitting Lemma, for any $\varphi \in \operatorname{End}(M)$ there is an n such that $M = \operatorname{Ker} \varphi^n \oplus \operatorname{Im} \varphi^n$. But M is indecomposable, so either $\operatorname{Im} \varphi^n = M$ or $\operatorname{Im} \varphi^n = 0$. In the first case φ^n, and then φ is also an epimorphism, consequently φ is an isomorphism, since $\dim_K M < \infty$. In the second case φ is nilpotent. This means that every element of R is either invertible or nilpotent.

We prove now that if a ring R has only invertible and nilpotent elements, then the nilpotent elements form a unique maximal (left/right) ideal.

Let N be the set of nilpotent elements. If $x \in N$ and $r \in R$, then xr and rx are zero divisors, so they cannot be invertible, thus $xr, rx \in N$.

If $x, y \in N$, then $x + y$ cannot be invertible, since in that case $xr + yr = 1$ for some r, where xr and yr are nilpotent, and the sum of two nilpotent elements cannot be 1, as we have seen in the proof of the locality of the indecomposable projective modules. So $x + y \in N$. Any proper right or left ideal is included in N because they cannot contain invertible elements, so N is the largest of all proper one-sided or two-sided ideals. Thus N is the Jacobson radical of R, and every element of R/N is invertible.

4. *Consider the map $M \mapsto D(M) = \operatorname{Hom}_K(M, K)$ from the category* mod-A *to* A-mod *(or from* A-mod *to* mod-A*). Let D act on the morphisms so that for $\alpha : M \to N$, we have $D(\alpha) : D(N) \to D(M)$, where $\varphi D(\alpha) := \alpha\varphi$. Prove that this D defines a vector space isomorphism from $\operatorname{Hom}(M, N)$ to $\operatorname{Hom}(D(N), D(M))$ for any modules M, N, furthermore, $D(\alpha\beta) = D(\beta)D(\alpha)$ and $D(1_M) = 1_{D(M)}$ (so D is a contravariant functor from* mod-A *to* A-mod*).*

Solution: $D(\alpha)$ is a homomorphism: $(\varphi + \psi)D(\alpha) = \alpha(\varphi + \psi) = \alpha\varphi + \alpha\psi = \varphi D(\alpha) + \psi D(\alpha)$ and for any $r \in A$ and $x \in M$, $x((r\varphi)D(\alpha)) = (x\alpha)(r\varphi) = ((x\alpha)r)\varphi = ((xr)\alpha)\varphi = (xr)(\alpha\varphi) = (xr)(\varphi D(\alpha)) = x(r(\varphi D(\alpha)))$.

$D : \operatorname{Hom}(M, N) \to \operatorname{Hom}(D(N), D(M))$ is a vector space homomorphism: $\varphi D(\alpha + \beta) = (\alpha + \beta)\varphi = \alpha\varphi + \beta\varphi = \varphi D(\alpha) + \varphi D(\beta) = \varphi(D(\alpha) + D(\beta))$, and for any $c \in K$, $\varphi D(c\alpha) = (c\alpha)\varphi = c(\alpha\varphi) = c(\varphi D(\alpha)) = \varphi(cD(\alpha))$, and this homomorphism is injective: if $\alpha \neq 0$, then $\exists m \in M$ such that $m\alpha \neq 0$, so $\exists \varphi : N_K \to K_K$ such that $m\alpha\varphi \neq 0$, i.e. $m(\varphi D(\alpha)) \neq 0$, which implies that $D(\alpha) \neq 0$. Finally, we should observe that $\dim_K(\mathrm{Hom}(M,N)) = \dim_K M \cdot \dim_K N = \dim_K D(M) \cdot \dim_K D(N) = \dim_K \mathrm{Hom}(D(N), D(M))$, and both M and N are finite dimensional, so the injective homomorphism $D : \mathrm{Hom}(M,N) \to \mathrm{Hom}(D(N), D(M))$ is also bijective.

For every φ, $\varphi D(\alpha\beta) = (\alpha\beta)\varphi = \alpha(\beta\varphi) = \alpha(\varphi D(\beta)) = (\varphi D(\beta))D(\alpha) = \varphi(D(\beta)D(\alpha))$, so $D(\alpha\beta) = D(\beta)D(\alpha)$, furthermore, $\varphi D(1_M) = 1_M\varphi = \varphi$ for every φ, so $D(1_M) = 1_{D(M)}$.

5. *Prove that in the setting of problem 4, $D^2(M) \cong M$ for every module, and D maps projective modules to injective modules and injective modules to projective modules.*

Solution: The substitution map $M \to D^2(M)$, $m \mapsto \Phi_m$, where $\varphi\Phi_m = m\varphi$ is a module homomorphism:
for every $\varphi \in D(M)$, $\varphi\Phi_{m+m'} = (m+m')\varphi = m\varphi + m'\varphi = \varphi\Phi_m + \varphi\Phi_{m'} = \varphi(\Phi_m + \Phi_{m'})$, and for every $\varphi \in D(M)$ and $a \in A$, $\varphi\Phi_{ma} = (ma)\varphi = m(a\varphi) = (a\varphi)\Phi_m = \varphi(\Phi_m a)$. This homomorphism is injective: if $m \neq 0$, then $\exists\varphi \in D(M)$ such that $m\varphi \neq 0$, i.e. $\varphi\Phi_m \neq 0$, so $\Phi_m \neq 0$. On the other hand, $\dim_K D^2(M) = \dim_K D(M) = \dim_K M < \infty$, so the homomorphism $m \mapsto \Phi_m$ is also bijective.

To prove that projective modules are mapped to injective modules, consider the diagram below on the left, where $P \in$ mod-A is projective, and $X, Y \in A$-mod.

$$\begin{tikzcd} X \arrow[r, tail, "\alpha"] \arrow[d, "\beta"'] & Y \\ D(P) & \end{tikzcd} \quad \Rightarrow \quad \begin{tikzcd} D(X) & D(Y) \arrow[l, two heads, "D(\alpha)"'] \\ D^2(P) \arrow[u, "D(\beta)"] \arrow[ur, dashed, "\exists\gamma"'] & \end{tikzcd}$$

Now take the K-dual of the diagram as shown on the right. By the statement of HW1, $D(\alpha)$ is an epimorphism. As we just proved, $D^2(P) \cong P$, so $D^2(P)$ is projective, hence there exists a $\gamma : D^2(P) \to D(Y)$ which makes the diagram commutative. Since the map $D : \mathrm{Hom}(Y, D(P)) \to \mathrm{Hom}(D^2(P), D(Y))$ is surjective, there is a $\gamma' : Y \to D(P)$ such that $D(\gamma') = \gamma$. Also, $D(\alpha\gamma') = D(\gamma')D(\alpha) = \gamma D(\alpha) = D(\beta)$, and $D : \mathrm{Hom}(X, D(P)) \to \mathrm{Hom}(D^2(P), D(X))$ is injective, so $\alpha\gamma' = \beta$, completing the diagram for injectivity.

With the (categorically) dual diagrams we get that the functor D maps injective modules to projective modules. It is also stated in HW1 that indecomposable modules are mapped to indecomposable ones, so the images of indecomposable projective modules are indecomposable injective modules, and we get every indecomposable injective module this way, since for any indec. injective $Q \in A$-mod, $D(Q)$ is an indec. projective in mod-A, and its image $D(D(Q)) \cong Q$.

6. *Suppose that A is a graph algebra with arrows $\alpha_1, \ldots, \alpha_k$, and that the module $M \in$ mod-A has a basis $\mathcal{B} = \{ b_1, \ldots, b_\ell \}$ such that the arrows map the set $\mathcal{B} \cup \{ 0 \}$ into itself. Let us denote by $b_i\alpha^{-1}$ the set $\{ x \in \mathcal{B} \mid x\alpha = b_i \}$. Prove that $\mathcal{B}' = \{ b_1', \ldots, b_k' \}$ is a basis of $D(M)$ if $b_i' : b_j \mapsto \delta_{ij}$, and for this basis $\alpha b_i' = (\sum b_i\alpha^{-1})'$.*

Solution: For any $\varphi \in D(M) = \mathrm{Hom}_K(M, K)$, $\varphi = \sum (b_i\varphi)b_i'$, so the elements b_i' form a generator set in $D(M)$, and they are clearly independent. For an arrow α, $b_j(\alpha b_i') = (b_j\alpha)b_i'$, which is 1 if $b_j\alpha = b_i$, and 0 otherwise. So $\alpha b_i' = \sum \{ b_j' \mid b_j\alpha = b_i \} = (\sum b_i\alpha^{-1})'$.

7. *Let $A_A = \begin{smallmatrix}1\\2\\2\end{smallmatrix} \oplus \begin{smallmatrix}2\\2\\2\end{smallmatrix}$. Determine the Loewy diagram of ${}_AA$, and by using the K-dual, determine all indecomposable injective right modules of A. Find the irreducible morphisms in* ind-A *going to projective or from injective modules.*

Solution: The graph of the algebra is Γ:

$$\begin{tikzcd} \overset{1}{\bullet} \arrow[r, "\alpha"] & \overset{2}{\bullet} \arrow[loop right, "\beta"] \end{tikzcd}$$

and $A \cong K\Gamma/I$, where $I = (\alpha\beta^2, \beta^3)$. The Loewy diagram of ${}_AA$ is the same as the Loewy diagram of $A'_{A'}$, where A' is the opposite algebra of A (meaning that the base set is the same but the product is in reverse order). The graph of A' is Γ':

$$\begin{tikzcd} \overset{1}{\bullet} & \overset{2}{\bullet} \arrow[l, "\alpha'"'] \arrow[loop right, "\beta'"] \end{tikzcd}$$

and $I' = ((\beta')^2\alpha', (\beta')^3)$. From this

$${}_AA = A'_{A'} = \begin{smallmatrix} 1 \end{smallmatrix} \oplus \begin{smallmatrix} & 2 & & \\ 1 & & 2 & \\ & & 1 & 2 \end{smallmatrix}.$$

The conditions of problem 6 hold for this module, and every element of the basis has only one inverse image for every arrow, so we get the Loewy diagram of the dual by turning the original diagrams upside down:

$$D({}_AA) = \begin{smallmatrix} 1 \end{smallmatrix} \oplus \begin{smallmatrix} & & 1 & 2 \\ 1 & & 2 & \\ & 2 & & \end{smallmatrix}.$$

The irreducible morphisms going to indecomposable projective modules are the embeddings of the direct summands of their radicals: $\begin{smallmatrix} 2 \\ 2 \end{smallmatrix} \to \begin{smallmatrix} 1 \\ 2 \\ 2 \end{smallmatrix}$ and $\begin{smallmatrix} 2 \\ 2 \end{smallmatrix} \to \begin{smallmatrix} 2 \\ 2 \\ 2 \end{smallmatrix}$. The irreducible morphisms going from indecomposable injective modules are the projections of the factors by the socles to their direct components: $\begin{smallmatrix} & 1 & 2 \\ 1 & & 2 \\ & 2 & \end{smallmatrix} \to 1$ és $\begin{smallmatrix} & 1 & 2 \\ 1 & & 2 \\ & 2 & \end{smallmatrix} \to \begin{smallmatrix} 1 & 2 \\ & 2 \end{smallmatrix}$.

8. *Find the Loewy diagram of the K-dual of the modules* $\begin{smallmatrix} & 1 \\ 1 & 2 \\ & 2 \end{smallmatrix}$ *and* $\begin{smallmatrix} 1 & 2 \\ 2 & 2 \\ & 2 \end{smallmatrix}$ *over the algebra of problem 7.*

Solution: Both modules satisfy the conditions of Problem 6. In the first, every basis element has at most one inverse image for every arrow, so the Loewy diagram of the dual module is the same as the original turned upside down: $\begin{smallmatrix} & 2 \\ 1 & 2 \\ & 1 \end{smallmatrix}$. If the basis elements of the second module are b, c_1, c_2, c_3, c_4, where

$$\begin{tikzcd}[column sep=small] b \arrow[d, dash, "\alpha"'] & & c_1 \arrow[d, dash, "\beta"] \\ c_2 \arrow[dr, dash, "\beta"'] & & c_3 \arrow[dl, dash, "\beta"] \\ & c_4 & \end{tikzcd}$$

then a basis of its dual is $\{\, b', c_1', c_2', c_3', c_4' \,\}$:

$$\begin{tikzcd}[column sep=small] & c_4' \arrow[dl, dash, "\beta'"'] & \\ c_2' \arrow[rr, dash, dashed] \arrow[d, dash, "\alpha'"'] & & c_3' \arrow[d, dash, "\beta'"] \\ b' & & c_1' \end{tikzcd}$$

i.e. $\beta' : c_4' \mapsto c_2' + c_3'$, $\beta' : c_3' \mapsto c_1'$ and $\alpha' : c_2' \mapsto b'$. Let $c_2'' = c_2' + c_3'$, then the Loewy diagram in the basis $\{\, b', c_1', c_2'', c_3', c_4' \,\}$ is:

$$\begin{tikzcd}[column sep=small] & c_4' \arrow[d, dash, "\beta'"'] & & \\ & c_2'' \arrow[dl, dash, "\alpha'"'] \arrow[dr, dash, "\beta'"] & & c_3' \arrow[dl, dash, "\beta'"] \\ b' & & c_1' & \end{tikzcd} \qquad \text{i.e.} \qquad \begin{tikzcd}[column sep=small] & 2 \arrow[d, dash] & & \\ & 2 \arrow[dl, dash] \arrow[dr, dash] & & 2 \arrow[dl, dash] \\ 1 & & 2 & \end{tikzcd}$$

Hf1. *Prove that the functor D on finite dimensional A-modules maps injective morphisms to surjective morphisms, surjective morphisms to injective morphisms, and $D(M)$ is indecomposable if and only if M is indecomposable. (Remember problem 2/2.)*

Hf2. *Determine the Loewy diagrams of the indecomposable injective modules of the algebra* $A_A = \begin{smallmatrix} 1 \\ 2 \\ 3 \end{smallmatrix} \oplus \begin{smallmatrix} & 2 & \\ 1 & & 3 \end{smallmatrix} \oplus \begin{smallmatrix} 3 \\ 1 \end{smallmatrix}$.