1. Theorem: *Let $0 \to X \to Y \to Z \to 0$ be an Auslander–Reiten sequence.*

1) There is an irreducible morphism $Z' \to Z$ if and only if $Z' \overset{\oplus}{\leq} Y$;

2) There is an irreducible morphism $X \to X'$ if and only if $X' \overset{\oplus}{\leq} Y$.

Prove the 'only if ' direction in both statements.

Solution: Let the maps in the ARS be $0 \to X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \to 0$.

1) Suppose that $Z' \xrightarrow{\gamma} Z$ is an irreducible morphism. Then γ cannot be a split epimorphism, so there must exist a morphism $Z' \xrightarrow{\delta} Y$ such that $\delta\beta = \gamma$. Since γ is irreducible, and β is not a split epimorphism, δ must be a split monomorphism, i.e. Z' is isomorphic to a direct summand of Y.

2) Similarly, if $X \xrightarrow{\gamma} X'$ is an irreducible morphism, then γ cannot be a split monomorphism, so there exists a morphism $Y \xrightarrow{\delta} X'$ such that $\alpha\delta = \gamma$. Since γ is irreducible, and α is not a split monomorphism, δ must be a split epimorphism. This implies that X' is isomorphic to a direct summand of Y.

2. *Let $A_A = P_1 \oplus P_2 \oplus \ldots \oplus P_n$ be a decomposition into indecomposable projective modules. We define a graph on $\mathcal{P} = \{ P_1, \ldots, P_n \}$ so that P_i and P_j are connected with an edge if and only if* $\mathrm{Hom}(P_i, P_j)$ *or* $\mathrm{Hom}(P_j, P_i)$ *is nonzero. Let $\mathcal{K}_1, \ldots, \mathcal{K}_t$ be the connected components of this graph. Prove that every $R_j := \oplus \{ P_i \,|\, P_i \in \mathcal{K}_j \}$ is an idecomposable ideal of A, so A is connected if and only if the graph on $\mathcal{P}$ is connected. In particular, a graph algebra $K\Gamma/I$ is connected if and only if Γ is connected.*

Solution: If $P \in \mathcal{K}_j$ és $r \in R$, then $rP = 1 \cdot rP = (e_1 + \ldots + e_n)rP = \oplus\, e_i rP$, where $P_i = e_i R$, so $e_i rP \leq P_i$ for every i. If $e_i rP \neq 0$, then $\mathrm{Hom}(P, P_i) \neq 0$ (the left multiplication by $e_i r$ gives such a homomorphism), so $P_i \in \mathcal{K}_j$. Thus $rP \leq R_j$ for every $P \in \mathcal{K}_j$ and $r \in R$, hence $R_i \triangleleft R$.

If $R_j = S \oplus T$ is a nontrivial direct sum of rings, then by the Krull–Schmidt Theorem, S and T are direct sums of projective modules from $\mathcal{K}_j$, thus there must be a homomorphism between some components of S and T, and this can be exended natually to a morphism between S and T. But $S^2 = S$ and $TS \leq T \cap S = 0$ imply that for $\varphi \in \mathrm{Hom}(S, T)$, we have $S\varphi = S^2\varphi = S\varphi S \leq TS = 0$, so $\mathrm{Hom}(S, T) = 0$, and similarly, $\mathrm{Hom}(T, S) = 0$, contradicting the assumptions.

Now suppose that A is a graph algebra with a graph Γ. If α is an arrow from i to j in Γ, then the left multiplication by α maps $e_j A$ nontrivially to $e_i A$. On the other hand, if there is a nontrivial homomorphism from $e_j A$ to $e_i A$, then it maps e_j to a linear combination of oriented paths from i to j. So the components of the graph of the algebra correspond to the components of the graph of projective modules defined in this problem.

3. *Let A be a graph algebra such that $A_A = \begin{smallmatrix} 1 \\ 3 \\ 1 \end{smallmatrix} \oplus \begin{smallmatrix} 2 \\ 1 \end{smallmatrix} \oplus \begin{smallmatrix} 3 \\ 1 \end{smallmatrix}$. Determine the Auslander–Reiten translate of the simple modules.*

Solution: We calculate the AR translate in the following way. The first two steps of the minimal projective resolution are:

$$P_1 \xrightarrow{\varphi} P_0 \longrightarrow M \longrightarrow 0.$$

(P_0 is the projective cover of M, i.e. $P_0 \twoheadrightarrow P_0/\operatorname{rad} P_0 \cong M/\operatorname{rad} M$, and $P_0 \to M$ is the map completing the diagram of projectivity for P_0. Then P_1 is the projective cover of the kernel.) Take the map φ^* between the corresponding left projective modules P_0' and P_1': If the components of the images of the generator elements of the components of P_1 are given in the row of a matrix as a vector of paths then the matrix for φ^* is the transpose of the matrix for φ, where we replace each path with its reverse in Γ' (actually, we apply the functor $\mathrm{Hom}(-, A_A)$): $P_0' \xrightarrow{\varphi^*} P_1'$, and complete this diagram with the cokernel of φ^*.

$$P_0' \xrightarrow{\varphi^*} P_1' \longrightarrow Tr(M) \longrightarrow 0,$$

Now $\tau(M) = D(Tr(M))$.

$A_A = \begin{smallmatrix}1\\3\\1\end{smallmatrix} \oplus \begin{smallmatrix}2\\1\end{smallmatrix} \oplus \begin{smallmatrix}3\\1\end{smallmatrix}$ $\qquad$ ${}_AA = \begin{smallmatrix}&1&\\2&&3\\&&1\end{smallmatrix} \oplus 2 \oplus \begin{smallmatrix}3\\1\end{smallmatrix}$

$\Gamma:\ 2 \xrightarrow{\alpha} 1 \underset{\gamma}{\overset{\beta}{\rightleftarrows}} 3$ $\qquad$ $\Gamma:\ 2 \xleftarrow{\alpha'} 1 \underset{\beta'}{\overset{\gamma'}{\rightleftarrows}} 3$

$\begin{smallmatrix}3\\1\end{smallmatrix} \xrightarrow{[\beta]} \begin{smallmatrix}1\\3\\1\end{smallmatrix} \to 1 \to 0$ $\qquad$ $\begin{smallmatrix}&1&\\2&&3\\&&1\end{smallmatrix} \xrightarrow{[\beta']} \begin{smallmatrix}3\\1\end{smallmatrix} \to 3 \to 0$ $\qquad$ $\tau(1) = D(3) = 3$

$\begin{smallmatrix}1\\3\\1\end{smallmatrix} \xrightarrow{[\alpha]} \begin{smallmatrix}2\\1\end{smallmatrix} \to 2 \to 0$ $\qquad$ $2 \xrightarrow{[\alpha']} \begin{smallmatrix}&1&\\2&&3\\&&1\end{smallmatrix} \to \begin{smallmatrix}1\\3\\1\end{smallmatrix} \to 0$ $\qquad$ $\tau(2) = D\left(\begin{smallmatrix}1\\3\\1\end{smallmatrix}\right) = \begin{smallmatrix}1\\3\\1\end{smallmatrix}$

$\begin{smallmatrix}1\\3\\1\end{smallmatrix} \xrightarrow{[\gamma]} \begin{smallmatrix}3\\1\end{smallmatrix} \to 3 \to 0$ $\qquad$ $\begin{smallmatrix}3\\1\end{smallmatrix} \xrightarrow{[\gamma']} \begin{smallmatrix}&1&\\2&&3\\&&1\end{smallmatrix} \to \begin{smallmatrix}1\\2\end{smallmatrix} \to 0$ $\qquad$ $\tau(3) = D\left(\begin{smallmatrix}1\\2\end{smallmatrix}\right) = \begin{smallmatrix}2\\1\end{smallmatrix}$

4. *Determine the Auslander–Reiten graph of the following graph algebras.*

a) $A = K\Gamma$, *where* $\Gamma : 1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3 \xleftarrow{\gamma} 4.$

b) $A_A = \begin{smallmatrix}1\\2\end{smallmatrix} \oplus \begin{smallmatrix}2\\2\end{smallmatrix}$

Solution: a) We calculate first those orbits of τ that start with an injective module (there might also be infinite or cyclic orbits). The series of translates stops when we it reaches a projective module.

$A_A = \begin{smallmatrix}1\\2\\3\end{smallmatrix} \oplus \begin{smallmatrix}2\\3\end{smallmatrix} \oplus 3 \oplus \begin{smallmatrix}4\\3\end{smallmatrix},$ $\qquad$ ${}_AA = 1 \oplus \begin{smallmatrix}2\\1\end{smallmatrix} \oplus \begin{smallmatrix}&3&\\2&&4\\1&&\end{smallmatrix} \oplus 4,$ $\qquad$ $D({}_AA) = 1 \oplus \begin{smallmatrix}1\\2\end{smallmatrix} \oplus \begin{smallmatrix}1&&\\2&&4\\&3&\end{smallmatrix} \oplus 4$

$\Gamma : 1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3 \xleftarrow{\gamma} 4$ $\qquad$ $\Gamma' : 1 \xleftarrow{\alpha'} 2 \xleftarrow{\beta'} 3 \xrightarrow{\gamma'} 4$

$\begin{smallmatrix}2\\3\end{smallmatrix} \xrightarrow{[\alpha]} \begin{smallmatrix}1\\2\\3\end{smallmatrix} \to 1 \to 0$ $\qquad$ $1 \xrightarrow{[\alpha']} \begin{smallmatrix}2\\1\end{smallmatrix} \to 2 \to 0$ $\qquad$ $\tau(1) = D(2) = 2$

$3 \xrightarrow{[\beta]} \begin{smallmatrix}2\\3\end{smallmatrix} \to 2 \to 0$ $\qquad$ $\begin{smallmatrix}2\\1\end{smallmatrix} \xrightarrow{[\beta']} \begin{smallmatrix}&3&\\2&&4\\1&&\end{smallmatrix} \to \begin{smallmatrix}3\\4\end{smallmatrix} \to 0$ $\qquad$ $\tau(2) = D(\begin{smallmatrix}3\\4\end{smallmatrix}) = \begin{smallmatrix}4\\3\end{smallmatrix}$

So the orbit of $Q(1)$ is

$$\left[\begin{smallmatrix}4\\3\end{smallmatrix} \;-\,-\,-\; 2 \;-\,-\,-\; 1\right].$$

$3 \xrightarrow{[\alpha\beta]} \begin{smallmatrix}1\\2\\3\end{smallmatrix} \to \begin{smallmatrix}1\\2\end{smallmatrix} \to 0$ $\qquad$ $1 \xrightarrow{[\beta'\alpha']} \begin{smallmatrix}&3&\\2&&4\\1&&\end{smallmatrix} \to \begin{smallmatrix}&3&\\2&&4\end{smallmatrix} \to 0$ $\qquad$ $\tau(\begin{smallmatrix}1\\2\end{smallmatrix}) = D(\begin{smallmatrix}&3&\\2&&4\end{smallmatrix}) = \begin{smallmatrix}2&&4\\&3&\end{smallmatrix}$

$3 \xrightarrow{[\beta\ \gamma]} \begin{smallmatrix}2\\3\end{smallmatrix} \oplus \begin{smallmatrix}4\\3\end{smallmatrix} \to \begin{smallmatrix}2&&4\\&3&\end{smallmatrix} \to 0$ $\qquad$ $\begin{smallmatrix}2\\1\end{smallmatrix} \oplus 4 \xrightarrow{\begin{bmatrix}\beta'\\ \gamma'\end{bmatrix}} \begin{smallmatrix}&3&\\2&&4\\1&&\end{smallmatrix} \to 3 \to 0$ $\qquad$ $\tau(\begin{smallmatrix}2&&4\\&3&\end{smallmatrix}) = D(3) = 3$

The orbit of $Q(2)$ is

$$\left[3 \;-\,-\,-\; \begin{smallmatrix}2&&4\\&3&\end{smallmatrix} \;-\,-\,-\; \begin{smallmatrix}1\\2\end{smallmatrix}\right].$$

$3 \xrightarrow{[\alpha\beta\ \gamma]} \begin{smallmatrix}1\\2\\3\end{smallmatrix} \oplus \begin{smallmatrix}4\\3\end{smallmatrix} \to \begin{smallmatrix}1&&\\2&&4\\&3&\end{smallmatrix} \to 0$ $\qquad$ $1 \oplus 4 \xrightarrow{\begin{bmatrix}\beta'\alpha'\\ \gamma'\end{bmatrix}} \begin{smallmatrix}&3&\\2&&4\\1&&\end{smallmatrix} \to \begin{smallmatrix}3\\2\end{smallmatrix} \to 0$ $\qquad$ $\tau(\begin{smallmatrix}1&&\\2&&4\\&3&\end{smallmatrix}) = D(\begin{smallmatrix}3\\2\end{smallmatrix}) = \begin{smallmatrix}2\\3\end{smallmatrix}$

The orbit of $Q(3)$ is

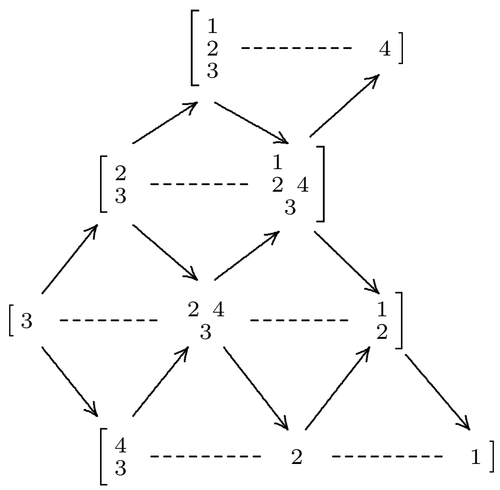
$$\left[\begin{smallmatrix}2\\3\end{smallmatrix} \;---\; \begin{smallmatrix}&1&\\2&&4\\&3&\end{smallmatrix}\right].$$

$3 \xrightarrow{[\gamma]} \begin{smallmatrix}4\\3\end{smallmatrix} \to 4 \to 0$ $\qquad$ $4 \xrightarrow{[\gamma']} \begin{smallmatrix}&3&\\2&&4\\&1&\end{smallmatrix} \to \begin{smallmatrix}3\\2\\1\end{smallmatrix} \to 0$ $\qquad$ $\tau(4) = D\left(\begin{smallmatrix}3\\2\\1\end{smallmatrix}\right) = \begin{smallmatrix}1\\2\\3\end{smallmatrix}$

The orbit of $Q(4)$ is

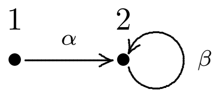
$$\left[\begin{smallmatrix}1\\2\\3\end{smallmatrix} \;---\; 4\right].$$

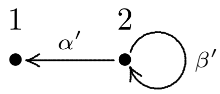
Knowing the irreducible morphisms going to projective and from injective modules, there is only one way to put together these orbits in the Auslander–Reiten graph. (Start with the irreducible morphisms going from 3 to $\begin{smallmatrix}2\\3\end{smallmatrix}$ and $\begin{smallmatrix}4\\3\end{smallmatrix}$, and from $\begin{smallmatrix}2\\3\end{smallmatrix}$ to $\begin{smallmatrix}1\\2\\3\end{smallmatrix}$, and use the fact that there is an arrow from $\tau(M)$ to N if and only if there is an arrow from N to M.) It is easy to check that this subgraph is a whole connected component: all the arrows going to projectives and injectives are there, and also the whole AR sequences for every module. Since this component is finite, this must be the whole Auslander–Reiten graph.



b)

$A_A = \begin{smallmatrix}1\\2\end{smallmatrix} \oplus \begin{smallmatrix}2\\2\end{smallmatrix}$, $\qquad$ ${}_AA = 1 \oplus \begin{smallmatrix}&2&\\1&&2\end{smallmatrix}$, $\qquad$ $D({}_AA) = 1 \oplus \begin{smallmatrix}1&&2\\&2&\end{smallmatrix}$

Γ: 

Γ': 

$\begin{smallmatrix}2\\2\end{smallmatrix} \xrightarrow{[\alpha]} \begin{smallmatrix}1\\2\end{smallmatrix} \to 1 \to 0$ $\qquad$ $1 \xrightarrow{[\alpha']} \begin{smallmatrix}&2&\\1&&2\end{smallmatrix} \to \begin{smallmatrix}2\\2\end{smallmatrix} \to 0$ $\qquad$ $\tau(1) = D\left(\begin{smallmatrix}2\\2\end{smallmatrix}\right) = \begin{smallmatrix}2\\2\end{smallmatrix}$

So the orbit of $Q(1)$ is

$$\left[\begin{smallmatrix}2\\2\end{smallmatrix} \;---\; 1\right].$$

$\begin{smallmatrix}2\\2\end{smallmatrix} \xrightarrow{[\alpha\ \beta]} \begin{smallmatrix}1\\2\end{smallmatrix} \oplus \begin{smallmatrix}2\\2\end{smallmatrix} \to \begin{smallmatrix}1&&2\\&2&\end{smallmatrix} \to 0$ $\qquad$ $1 \oplus \begin{smallmatrix}&2&\\1&&2\end{smallmatrix} \xrightarrow{\left[\begin{smallmatrix}\alpha'\\\beta'\end{smallmatrix}\right]} \begin{smallmatrix}&2&\\1&&2\end{smallmatrix} \to 2 \to 0$ $\qquad$ $\tau\left(\begin{smallmatrix}1&&2\\&2&\end{smallmatrix}\right) = D(2) = 2$

$\begin{smallmatrix}2\\2\end{smallmatrix} \xrightarrow{[\beta]} \begin{smallmatrix}2\\2\end{smallmatrix} \to 2 \to 0$ $\qquad$ $\begin{smallmatrix}&2&\\1&&2\end{smallmatrix} \xrightarrow{[\beta']} \begin{smallmatrix}&2&\\1&&2\end{smallmatrix} \to \begin{smallmatrix}2\\1\end{smallmatrix} \to 0$ $\qquad$ $\tau(2) = D\left(\begin{smallmatrix}2\\1\end{smallmatrix}\right) = \begin{smallmatrix}1\\2\end{smallmatrix}$

So the orbit of $Q(2)$ is

$$\left[\begin{smallmatrix}1\\2\end{smallmatrix} - - - \;2\; - - - \begin{smallmatrix}1&&2\\&2\end{smallmatrix}\right].$$

We know that there are arrows from 2 to $\begin{smallmatrix}1\\2\end{smallmatrix}$ and $\begin{smallmatrix}2\\2\end{smallmatrix}$, and the inverse translate of 2 is $\begin{smallmatrix}1&&2\\&2\end{smallmatrix}$. These give the whole ARS starting at 2, since $\dim \begin{smallmatrix}1\\2\end{smallmatrix} + \dim \begin{smallmatrix}2\\2\end{smallmatrix} = \dim 2 + \dim \begin{smallmatrix}1&&2\\&2\end{smallmatrix}$. If we continue this with the inverse translates, we get the following graph.

$$\begin{tikzcd}
 & {\left[\begin{smallmatrix}1\\2\end{smallmatrix}\right.} \arrow[rr, dash, dashed] \arrow[dr] & & {\left.2\right]} \\
{-\,-\,-\;2} \arrow[ur] \arrow[dr] \arrow[rr, dash, dashed] & & {\left.\begin{smallmatrix}1&&2\\&2\end{smallmatrix}\right]} \arrow[ur] \arrow[dr] & \\
 & {\left[\begin{smallmatrix}2\\2\end{smallmatrix}\right.} \arrow[ur] \arrow[rr, dash, dashed] & & {\left.1\right]}
\end{tikzcd}$$

Here we repeat the module 2, to show that both the ARS starting at 2 and the ARS ending at 2 are complete. It can be checked easily that this is a complete component of the Auslander–Reiten graph: all the irreducible morphisms going to projectives and going from injectives are there, and also all the arrows given by the AR sequences (one has to check the dimensions to see that no middle terms are missing). So there are only five indecomposable modules in mod-A: 1, 2, $\begin{smallmatrix}1\\2\end{smallmatrix}$, $\begin{smallmatrix}2\\2\end{smallmatrix}$ and $\begin{smallmatrix}1&&2\\&2\end{smallmatrix}$.

5. *Prove that $Z(M_n(R)) = Z(R)I_n$, where I_n denotes the $n \times n$ identity matrix.*

Solution: Suppose a matrix M is in the center. Then for every $i \neq j$, $m_{ij}E_{ii} = E_{ii}ME_{ji} = E_{ii}E_{ji}M = 0M = 0$, so $m_{ij} = 0$, furthermore $m_{ii}E_{ij} = E_{ii}ME_{ij} = E_{ii}ME_{ij}E_{jj} = E_{ii}E_{ij}ME_{jj} = E_{ij}ME_{jj} = m_{jj}E_{ij}$, so $m_{ii} = m_{jj}$. This means, that $M = mI$ for some $m \in R$. But $rmI_n = (rI_n)M = M(rI_n) = mrI_n$, so $m \in Z(R)$. Finally, it is obvious that $Z(R)I_n \leq Z(M_n(R))$, since for any matrix M and $r \in Z(R)$, $(rI_n)M = [rm_{ij}] = [m_{ij}r] = M(rI_n)$.

6. *Let $S = eR \overset{\oplus}{\leq} R$ be a simple module generated by the idempotent element e. Prove that* $\operatorname{End} S \cong eRe$. *In particular, if R is a full matrix ring over a division ring D then* $\operatorname{End} S \cong D$.

Solution: Here the endomorphisms are supposed to act from the left, otherwise $\operatorname{End} S$ would be isomorphic to the opposite ring of eRe.
For every element $a := ere \in eRe$, the left multiplication φ_a by a gives a module homomorphism from eR to eR. The map $\varphi : eRe \to \operatorname{End}(S)$, $\varphi : a \mapsto \varphi_a$ is a ring homomorphism, since $(\varphi_a + \varphi_b)s = \varphi_a s + \varphi_b s = as + bs = (a+b)s = \varphi_{a+b}s$ and $(\varphi_a\varphi_b)s = \varphi_a(\varphi_b s) = \varphi_a(bs) = a(bs) = (ab)s = \varphi_{ab}s$. The map φ is injective because for $a \neq 0$, $\varphi_a e = ae = eree = ere =\neq 0$, so $\varphi_a \neq 0$. Finally, φ is surjective because for any endomorphism $\alpha \in \operatorname{End} S$ and $a := \alpha e$, we have $\alpha(er) = \alpha(eer) = \alpha(e)er = a(er) = \varphi_a(er)$ for every $r \in R$, so $\alpha = \varphi_a$.
The full matrix ring has only one simple module up to isomorphism, one copy of this is $S = E_{11}M_n(D)$. So the endomorphism ring of S is isomorphic to $E_{11}M_n(D)E_{11} = DE_{11} \cong D$.

7. *Find the irreducible representations of C_3 over an arbitrary field K. Determine the submodules of KC_3 when* $\operatorname{char} K = 3$.

Solution: The representations of C_3 are the group homomorphisms going to $GL(V)$. The representation is determined by the image of the generator element: a matrix A such that $A^3 = I$. The representation is irreducible if V has no proper invariant subspace. Let $m(x)$ be the minimal polinomial of A. $A^3 = I$ implies that $m(x) \mid x^3 - 1$. If $m(1) = 0$, then 1 is an eigenvalue of A, so A has an eigenvector, and the generated subspace is A-invariant, thus $\dim V = 1$, and A is the identity of $GL(V)$. This gives the trivial representation of C_3. If $\operatorname{char} K = 3$, then $x^3 - 1 = (x-1)^3$, so in this case there is no other irreducible representation..

Suppose now that $\operatorname{char} K \neq 3$ and $m(1) \neq 0$. Then $m(x) \mid x^2 + x + 1$. If $x^2 + x + 1$ is reducible over K, then it has two different roots ($x^3 - 1$ has no multiple roots when $\operatorname{char} K \neq 3$, since $x^3 - 1$ is prime to its derivative), and there is a one-dimensional representation for each eigenvalue. In this case C_3 has three different (and clearly non-equivalent) representations.

If $x^2 + x + 1$ is irredubible over K, then for every $v \in V$, the subspace generated by v and vA is A-invariant (since $vA^2 = -v - vA$), and it has no proper A-invariant subspace because then A would have an eigenvector. már nincs A-invariáns altere, mert akkor A-nak lenne sajátvektora. So V must be two-dimensional. Such a representation exists: $A = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}$. There cannot be any more non-equivalent irreducible representations because $1 + 2 = \dim \mathbb{C}C_3$.

If $\operatorname{char} K = 3$, then we saw that there is only one simple module up to isomorphism, let's call it S. Then every minimal submodule of KC_3 must be isomorphic S, so it is one-dimensional, and for the generator element $x + ya + za^2$ (where $C_3 = \langle a \rangle$), satisfies $(x + ya + za^2)a = z + xa + ya^2 = x + ya + za^2$, hence $x = y = z$.This means that $M_1 = \{ \lambda(1 + a + a^2) \mid \lambda \in K \}$ is the only minimal submodule in KC_3-ban, thus every other nontrivial module contains it. On the other hand, for every maximal submodule M_2, the factor modules $M_2/M_1 \cong S$, so for every $u \in M_2$, $ua - u \in M_1$, and these are exactly the elements $u = x + ya + za^2$, for which $x + y + z = 0$. Since these elements form a 2-dimensional submodule the maximal submodule is also unique. Thus KC_3 has only four submodules: $0, M_1, M_2, KC_3$.

8. *Find the irreducible representations of $C_2 \times C_2$ over an arbitrary field K.*

Solution: The homomorphisms from $C_2 \times C_2$ to $GL(V)$ are determined by the images of the generator elements, i.e. a pair of matrices A and B such that $A^2 = B^2 = I$ and $AB = BA$. The minimal polynomial of A and B divides $x^2 - 1$, so A and B must have eigenvalues in K. Let V_1 be an eigenspace of A. Since $AB = BA$, this subspace is also B-invariant: if $vA = \lambda v$, then $(vB)A = v(BA) = v(AB) = (vA)B = \lambda vB$. B also has an eigenvector in this subspace because $B^2 = I$, so A and B has a common eigenvector, which means that V has a one-dimensional $C_2 \times C_2$-invariant subspace. So every irreducible representation is linear, i.e. it maps $C_2 \times C_2$ to $K^\times$. Both generators must be mapped to 1 or -1, and this way we always get a homomorphism, so in case $\operatorname{char} K = 2$ there is only one, in every other case there are four irreducible representations.

9. *Prove that* $J(KG) = \left\{ \sum_{g \in G} \lambda_g g \;\middle|\; \sum_{g \in G} \lambda_g = 0 \right\}$ *if G is a finite p-group and* $\operatorname{char} K = p$. *How many nonisomorphic simple modules exist in* mod-KG*?*

Solution: First we show by induction on $|G|$ that G has only one irreducible representation, the trivial one. Let $\varphi : G \to GL(V)$ be irreducible. Since G is a p-group, $Z(G) \neq 1$. Let $1 \neq g \in Z(G)$. If $|G| = p^n$, then $(\varphi(g))^{p^n} = \varphi(g^{p^n}) = \varphi(1) = I$, so the minimal polynomial of $\varphi(g)$ is a divisor of $x^{p^n} - 1 = (x-1)^{p^n}$ (the equality holds because $\operatorname{char} K = p$), hence $\varphi(g)$ has an eigenvector for eigenvalue 1. Let V_1 be the eignespace of $\varphi(g)$ for the eigenvalue 1. Since $g \in Z(G)$, this subspace is $\operatorname{Im}\varphi$-invariant, so the irreducibility of φ implies that $V_1 = V$, and $g \in \operatorname{Ker}\varphi$. But then $\bar{\varphi} : G/\langle g\rangle \to GL(V)$ is also an irreducible representation, and it is trivial by the induction hypothesis, thus φ is also trivial.

Let $A = KG$ and $M_0 = \{\sum\limits_{g\in G} \lambda_g g \mid \sum\limits_{g\in G} \lambda_g = 0\}$. M_0 is clearly a submodule of A_A, and $\dim_K M_0 = \dim A_K - 1$, so M_0 is maximal. On the other hand, for any maximal submodule M of A_A, the factor module A_A/M must be isomorphic to the only irreducible A-module, which is trivial, so in the factor module $1 \cdot g = 1$, i.e. $1 - g \in M$ for every $g \in G$. But these elements generate M_0, so $M_0 \leq M$, and then the maximality of M_0 implies that $M_0 = M$. This shows that the only maximal submodule of A_A is M_0, hence $J(A) = M_0$.

HW1. *Consider the graph algebra with Loewy diagram* $A_A = \begin{smallmatrix} & 1 & \\ 2 & & 3 \\ 1 & & \end{smallmatrix} \oplus \begin{smallmatrix} 2 \\ 1 \\ 2 \end{smallmatrix} \oplus \begin{smallmatrix} 3 \end{smallmatrix}$. *Calculate the AR translate of the module* $\begin{smallmatrix} 1 \\ 2 \\ 1 \end{smallmatrix}$.

HW2. *Prove that the nonzero morphism* $\begin{smallmatrix} 2 \\ 1 \end{smallmatrix} \to \begin{smallmatrix} 1 \\ 2 \\ 1 \end{smallmatrix}$ *is not an irreducible morphism. (Use the result of HW1, and show that the first module cannot be a direct summand of the middle term of the AR sequence, or give a proper decomposition of the morphism, and prove that it is proper.)*