1. *Find the nonlinear irreducible representation of the quaternion group over $\mathbb{C}$.*

Solution: We try to find a subgroup of $GL_2(\mathbb{C})$ isomorphic to Q. For this we look for two matrices B and C such that $B^2 = C^2 = -I$ and $B^{-1}CB = C^{-1} = -C$. The minimal polinomial of B and C must be x^2+1. We may choose B to be diagonal: $B = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$. If $C = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $B^{-1}CB = -C$ gives that $\begin{bmatrix} a & -b \\ -c & d \end{bmatrix} = \begin{bmatrix} -a & -b \\ -c & -d \end{bmatrix}$, i.e. $a = d = 0$, and $k_C(x) = x^2 - bc = x^2 + 1$. Let us choose $b = 1$ and $c = -1$, i.e. $C = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. The subgroup $\langle B, C \rangle$ is isomorphic to Q with the isomorphism mapping i to B and j to C.

2. *Prove that the product of an irreducible and a linear character is an irreducible character.*

Solution: Let λ be linear and χ an arbitrary irreducible character. Then $|\lambda(g)| = 1$ for every $g \in G$, so

$[\lambda\chi, \lambda\chi] = \frac{1}{|G|} \sum_{g \in G} \lambda(g)\chi(g)\overline{\lambda(g)\chi(g)} = \frac{1}{|G|} \sum_{g \in G} |\lambda(g)|^2 |\chi(g)|^2 = \frac{1}{|G|} \sum_{g \in G} |\chi(g)|^2 = [\chi, \chi] = 1,$

hence $\lambda\chi$ is irreducible.

3. *The orientation preserving isometries of the cube form a group isomorphic to S_4. Is this representation irreducible?*

Solution: Let the cube be $[-1,1] \times [-1,1] \times [-1,1]$. The group acts faithfully on the four diagonals of the cube. Let us call the diagonals starting at the corners $(-1,-1,-1), (1,-1,-1), (1,1,-1), (-1,1,-1)$ a, b, c, d. Let us calculate the values of the character χ corresponding to the given representation. Then the standard matrix of the rotation about the z axis by $+90°$ is

$$\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{yielding the permutation } (abcd) \quad \Rightarrow \quad \chi((....)) = 1.$$

The standard matrix of the rotation about the y axis by $180°$ is

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad \text{yielding the permutation } (ab)(cd) \quad \Rightarrow \quad \chi((..)(..)) = -1.$$

The product of these two is

$$\begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad \text{yielding the permutation } (bd) \quad \Rightarrow \quad \chi((..)) = -1.$$

The matrix of the rotation about a by $120°$ in the basis consisting of the vectors pointing from the origin to $(-1,1,1), (1,-1,1), (1,1,-1)$ is

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad \text{yielding the permutation } (bdc) \quad \Rightarrow \quad \chi((...)) = 0.$$

So the character of the representation is

	1	(..)(..)	(...)	(....)	(..)
χ	3	-1	0	1	-1

$[\chi,\chi] = \frac{1}{24}(1\cdot 3^2 + 3\cdot(-1)^2 + 8\cdot 0^2 + 6\cdot 1^2 + 6\cdot(-1)^2) = 1$, so the representation is irreducible.

4. *Determine the character table of A_4 and S_4.*

Solution: The conjugacy classes of A_4 are 1, (..)(..), $(...)_1 := (123)^{A_4}$ and $(...)_2 := (132)^{A_4}$, whose sizes are 1, 3, 4 and 4 (the conjugacy class of 3-cycles in S_4 breaks into two in A_4, since $C_{A_4}((abc)) = C_{S_4}((abc)) \Rightarrow |\mathcal{K}_{A_4}((abc))| = |A_4 : C_{A_4}((abc))| = \frac{1}{2}|S_4 : C_{S_4}((abc))| = \frac{1}{2}|\mathcal{K}_{S_4}((abc))|$).
$A_4/A_4' = A_4/\langle(..)(..)\rangle \cong C_3$, so A_4 has 3 linear characters, and these can be expanded from the character table of C_3.

C_3	1	a	a^2
χ_1	1	1	1
χ_2	1	ε	ε^2
χ_3	1	ε^2	ε

A_4	1	(..)(..)	$(...)_1$	$(...)_2$
χ_1	1	1	1	1
χ_2	1	1	ε	ε^2
χ_3	1	1	ε^2	ε
χ_4	3	-1	0	0

The fourth character of A_4 must be of degree 3 ($12 = 1^2+1^2+1^2+3^2$), and the other values can be calculated by the vertical orthogonality relation: we make each column orthogonal to the first.

The conjugacy classes of S_4 are 1, (..)(..), (...), (....), (..) with sizes $1, 3, 8, 6, 6$. For the normal subgroup $V = \langle(..)(..)\rangle$, the factor group $S_4/V \cong S_3$, since $S_3 \cap V = 1$ and $S_3V = S_4$. So we can obtain three irreducible characters from the character table of S_3, and the fourth from problem 3. Then $\chi_2\chi_4$ is also irreducible by Problem 2, and it is different from the other three, so $\chi_5 = \chi_2\chi_4$.

S_3	1	(...)	(..)
χ_1	1	1	1
χ_2	1	1	-1
χ_3	2	-1	0

S_4	1	(..)(..)	(...)	(....)	(..)
χ_1	1	1	1	1	1
χ_2	1	1	1	-1	-1
χ_3	2	2	-1	0	0
χ_4	3	-1	0	1	-1
χ_5	3	-1	0	-1	1

5. *Which of the following class functions of S_4 are characters? Write the characters as sums of irreducible characters.*

1	(..)(..)	(...)	(....)	(..)
5	2	-6	0	1
5	-1	$\frac{3}{2}$	1	0
8	0	-1	2	-2
4	0	-2	0	2

Solution: The first row cannot be a character because for every character χ and $g \in G$, the inequality $|\chi(g)| \leq \chi(1)$ must hold. The second row cannot be a character either, because $\frac{3}{2}$ is not an algebraic integer (its minimal polinomial over $\mathbb{Z}$ is $2x-3$, whose main coefficient is not a unit element). Let us call the third and the fourth class function χ and ψ. Calculate the coefficients of the irreducible characters χ_i $(i = 1, 2, 3, 4, 5)$ for the decomposition of χ and ψ.

$$\begin{aligned}
[\chi, \chi_1] =& \frac{1}{24}(1 \cdot 8 + 3 \cdot 0 + 8 \cdot (-1) + 6 \cdot 2 + 6 \cdot (-2)) = 0, \\
[\chi, \chi_2] =& 1 \cdot 8 + 3 \cdot 0 + 8 \cdot (-1) + 6 \cdot (-2) + 6 \cdot 2) = 0 \\
[\chi, \chi_3] =& \frac{1}{24}(1 \cdot 16 + 3 \cdot 0 + 8 \cdot 1 + 6 \cdot 0 + 6 \cdot 0 = 1, \\
[\chi, \chi_4] =& \frac{1}{24}(1 \cdot 24 + 3 \cdot 0 + 8 \cdot 0 + 6 \cdot 2 + 6 \cdot 2) = 2, \\
[\chi, \chi_5] =& \frac{1}{24}(1 \cdot 24 + 3 \cdot 0 + 8 \cdot 0 + 6 \cdot (-2) + 6 \cdot (-2)) = 0,
\end{aligned}$$

So $\chi = \chi_3 + 2\chi_4$ is a character, and it is not irreducible.

$$\begin{aligned}
[\psi, \chi_1] =& \frac{1}{24}(1 \cdot 4 + 3 \cdot 0 + 8 \cdot (-2) + 6 \cdot 0 + 6 \cdot 2) = 0, \\
[\psi, \chi_2] =& \frac{1}{24}(1 \cdot 4 + 3 \cdot 0 + 8 \cdot (-2) + 6 \cdot 0 + 6 \cdot (-2)) = -1,
\end{aligned}$$

This already shows that ψ cannot be a character because in its decomposition into a linear combination of irreducible characters there is a negative coefficient.

Hf1. *Let G be a group of order 28. Prove that G has an irreducible representation of degree 2 over $\mathbb{C}$.*

Hf2. *Complete the following table if we know that this is the character table of a finite group (the rows and columns are not necessarily in the usual order). What is the order of the* *group? Determine the sizes of its conjugacy classes and the orders of all normal subgroups.*

1		1	-1		1	
1					-1	1
		1	-1			1
			0	0		-2
2		-2	0		$-i\sqrt{2}$	0
2	0		0			0