**1.** *Prove the orbit-counting theorem (Burnside's lemma): if $\varphi : G \to S_\Omega$ is an action of the group $G$ on a set $\Omega$, then the number of orbits of $G$ in $\Omega$ is*

$$\frac{1}{|G|}\sum_{g\in G} |\operatorname{Fix}(g)|.$$

*Solution:* Consider the set $\mathcal{S} = \{(g,\omega) \mid g \in G,\ \omega \in \Omega,\ \omega g = \omega\}$. If we denote by $\mathcal{O}_1, \dots, \mathcal{O}_t$ the orbits of $G$, and $\omega_1, \dots, \omega_t$ are representatives of the orbits, then

$$|\mathcal{S}| = \sum_{g\in G} |\operatorname{Fix}(g)|, \text{ and}$$

$$|\mathcal{S}| = \sum_{\omega\in\Omega} |G_\omega| = \sum_{i=1}^{t} |\mathcal{O}_i| \cdot |G_{\omega_i}| = \sum_{i=1}^{t} |G| = |G|t,$$

so the number of orbits is $t = \frac{1}{|G|}\sum\limits_{g\in G} |\operatorname{Fix}(g)|$. (Here we used the fact that for two elements of the same orbit, $\omega$ and $\omega g$, $G_{\omega g} = g^{-1}G_\omega g$, so $|G_{\omega g}| = |G_\omega|$.)

**2.** *Prove that $1_G$ is always a summand in the decomposition of any permutation character into irreducibles. What is the coefficient of $1_G$?*

*Solution:* For a permutation character $\chi$, the scalar product $[\chi, 1_G] = \frac{1}{|G|}\sum\limits_{g\in G} \chi(g) \cdot 1 = \frac{1}{|G|}\sum\limits_{g\in G} |\operatorname{Fix}(g)|$, which is the number of orbits by the orbit-counting theorem. So it is at least 1.

**3.** *Prove that for the permutation character $\chi$ of a 2-transitive group action, $\chi - 1_G$ is always irreducible.*

*Solution:* $[\chi,\chi] = \frac{1}{|G|}\sum\limits_{g\in G} |\operatorname{Fix}(g)|^2 = \frac{1}{|G|}\sum\limits_{g\in G} |\{(\alpha,\beta) \mid \alpha,\beta \in \operatorname{Fix}(g)\}| =$
$= \frac{1}{|G|}\sum\limits_{g\in G} |\{(\alpha,\beta) \mid \alpha,\beta \in \operatorname{Fix}(g),\ \alpha \neq \beta\}| + \frac{1}{|G|}\sum\limits_{g\in G} |\{(\alpha,\alpha) \mid \alpha \in \operatorname{Fix}(g)\}| = 1 + 1 = 2$
by the orbit-counting theorem, because $G$ is 2-transitive, so it acts transitively both on $\Omega^{(2)} := \{(\alpha,\beta) \mid \alpha,\beta \in \Omega,\ \alpha \neq \beta\}$ and on $\Omega$. So if the decomposition of $\chi$ to a sum of irreducible characters is $\sum\limits_{i=1}^{k} a_i\chi_i$, then $2 = [\chi,\chi] = \sum\limits_{i=1}^{k} a_i^2$. Since the coefficients are nonnegative integers, this can only happen if there are two indices $i \neq j$ such that $a_i = a_j = 1$, and the others are 0. Finally, by Problem 2, $1_G$ is a summand of $\chi$, so $\chi - 1_G$ is the other irreducible component.

**4.** *Let $H \leq K \leq G$, and let $\varphi$ be a class function of $H$. Show that $(\varphi^K)^G = \varphi^G$.*

*Solution:*

$$(\varphi^K)^G(g) = \frac{1}{|K|}\sum_{x\in G}(\varphi^K)^\circ(xgx^{-1}) = \frac{1}{|K|}\sum_{\substack{x\in G\\ xgx^{-1}\in K}} (\varphi^K)(xgx^{-1}) =$$

$$\frac{1}{|K||H|}\sum_{\substack{x\in G\\ xgx^{-1}\in K}}\sum_{y\in K}\varphi^\circ(yxgx^{-1}y^{-1}) = \frac{1}{|K||H|}\sum_{y\in K}\sum_{\substack{x\in G\\ (yx)g(yx)^{-1}\in K}}\varphi^\circ(yxgx^{-1}y^{-1}) =$$

$$\frac{1}{|K||H|} \cdot |K| \sum_{z \in G} \varphi^{\circ}(zgz^{-1}) = \varphi^G(g).$$

Another solution, using the Frobenius reciprocity:
For every $\chi_i \in \operatorname{Irr} G$, we have $[(\varphi^K)^G, \chi_i] = [\varphi^K, (\chi_i)_K] = [\varphi, ((\chi_i)_K)_H] = [\varphi, (\chi_i)_H] = [\varphi^G, \chi_i]$, so in the decomposition of $(\varphi^K)^G$ and of $\varphi^G$ into a sum of irreducible characters, the coefficients are equal, hence $(\varphi^K)^G = \varphi^G$.

**5.** *What is the induced character to $A_4$ of*
*a) the trivial character of the Klein group $\langle (..)(..) \rangle$;*
*b) a nontrivial character of the cyclic group $\langle (123) \rangle$?*

*Solution:* a) $G = A_4$, $H = V = \langle (..)(..) \rangle$, $\chi = 1_V$

| | 1 | 1 | 1 | 1 |
|---|---|---|---|---|
| $H$ | 1 | (12)(34) | (13)(24) | (14)(23) |
| $\chi$ | 1 | 1 | 1 | 1 |

| | 1 | 3 | 4 | 4 |
|---|---|---|---|---|
| $G$ | 1 | (..)(..) | $(123)^G$ | $(132)^G$ |
| $\chi^G$ | 3 | 3 | 0 | 0 |

$\chi^G(1) = |G : H| = 3,$
$\chi^G((..)(..)) = 3(\frac{1}{3} \cdot 1 + \frac{1}{3} \cdot 1 + \frac{1}{3} \cdot 1) = 3$
$\chi^G((123)) = \chi^G((132)) = 0.$

b) $G = A_4$, $H = \langle (123) \rangle$. Since $H \cong C_3$ is abelian, the irreducible representations are all linear, and the image of the generator element must be a third root of unity. $\chi$ is nontrivial, so $\chi((123)) = \varepsilon$, where $\varepsilon$ is a primitive third root of unity.

| | 1 | 1 | 1 |
|---|---|---|---|
| $H$ | 1 | (123) | (132) |
| $\chi$ | 1 | $\varepsilon$ | $\varepsilon^2$ |

| | 1 | 3 | 4 | 4 |
|---|---|---|---|---|
| $G$ | 1 | (..)(..) | $(123)^G$ | $(132)^G$ |
| $\chi^G$ | 4 | 0 | $\varepsilon$ | $\varepsilon^2$ |

$\chi^G(1) = |G : H| = 4$
$\chi^G((..)(..)) = 0$
$\chi^G((123)) = 4(\frac{1}{4} \cdot \varepsilon) = \varepsilon$
$\chi^G((132)) = 4(\frac{1}{4} \cdot \varepsilon^2) = \varepsilon^2.$

**6.** *a) Consider the action of $S_4$ on the partitions of $\{1, 2, 3, 4\}$ to subsets of the following sizes: $4$, $3+1$, $2+2$, $2+1+1$, $1+1+1+1$, where the order of the subsets matters but the order of the elements within each subset does not. Calculate the permutation characters corresponding to these group actions, and write them as sums of irreducible characters.*

*b) Use the method of part a) to determine the character table of $S_5$: calculate the permutation characters obtained from the group action on the partitions to sizes $5$, $4+1$, $3+2$ and $3+1+1$ (the subsets are in decreasing order, and the partition types are taken in lexicographic order), and find a new irreducible character from each by subtracting the already known irreducible components. Instead of continuing with the remaining partitions, complete the table by multiplying the irreducible characters by the alternating linear character.*

*c) Restrict the irreducible characters of $S_5$ to $A_5$. Which of the restrictions are irreducible. Complete the character table of $A_5$ by taking the induced character of a nontrivial character of $\langle (12345) \rangle$ to $A_5$.*

*Solution:* a) Let us call the characters defined by the given types of partitions $\varphi_4$, $\varphi_{3+1}$, $\varphi_{2+2}$, $\varphi_{2+1+1}$ and $\varphi_{1+1+1+1}$. We have to determine the number of fixed partitions for every element of $S_4$ among the partitions of a given type. A partition is fixed by a permutation if and only if all of its cycles act within one class of the partition. The degree of each character is the number of partitions of the given type:
$\varphi_4(1) = \binom{4}{4} = 1$, $\varphi_{3+1}(1) = \binom{4}{3}\binom{1}{1} = 4$, $\varphi_{2+2}(1) = \binom{4}{2}\binom{2}{2} = 6$,
$\varphi_{2+1+1} = \binom{4}{2}\binom{2}{1}\binom{1}{1} = 12$, $\varphi_{1+1+1+1} = 4 \cdot 3 \cdot 2 \cdot 1 = 24$.
It is clear that $\varphi_4(g) = 1$ for every $g \in S_4$, and $\varphi_{1+1+1+1}(g) = 0$ for every $g \neq 1$.

| | 1 | 3 | 8 | 6 | 6 |
|---|---|---|---|---|---|
| | 1 | (..)(..) | (...) | (..) | (....) |
| $\varphi_4$ | 1 | 1 | 1 | 1 | 1 |
| $\varphi_{3+1}$ | 4 | 0 | 1 | 2 | 0 |
| $\varphi_{2+2}$ | 6 | 2 | 0 | 2 | 0 |
| $\varphi_{2+1+1}$ | 12 | 0 | 0 | 2 | 0 |
| $\varphi_{1+1+1+1}$ | 24 | 0 | 0 | 0 | 0 |

Here are two examples how we can calculate the values of the above characters. $\varphi_{3+1}((12)) = 2$ because in a fixed partition $\{1, 2\}$ must be in the first class, and either 3 or 4 is added to it. $\varphi_{2+1+1}((12)) = 2$ because $\{1, 2\}$ must be in first class, 3 or 4 in the second, and the remaining element in the last.
$S_4$ is clearly transitive on all the partitions of a given type, so $1_{S_4}$ is a summand with coefficient 1 in each of the five permutation characters. We can determine the other coefficients by taking the scalar products with $\chi_i$ $(i = 2, 3, 4, 5)$ in the character table below. We get that $\varphi_4 = \chi_1$, $\varphi_{3+1} = \chi_1 + \chi_4$, $\varphi_{2+2} = \chi_1 + \chi_3 + \chi_4$,
$\varphi_{2+1+1} = \chi_1 + \chi_3 + 2\chi_4 + \chi_5$, $\varphi_{1+1+1+1} = \chi_1 + \chi_2 + 2\chi_3 + 3\chi_4 + 3\chi_5 = \rho_{S_4}$.

b)

| | 1 | 15 | 20 | 24 | 10 | 20 | 30 |
|---|---|---|---|---|---|---|---|
| | 1 | (..)(..) | (...) | (.....) | (..) | (...)(..) | (....) |
| $\varphi_5$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $\varphi_{4+1}$ | 5 | 1 | 2 | 0 | 3 | 0 | 1 |
| $\varphi_{3+2}$ | 10 | 2 | 1 | 0 | 4 | 1 | 0 |
| $\varphi_{3+1+1}$ | 20 | 0 | 2 | 0 | 6 | 0 | 0 |

$\varphi_5 = \chi_1$ is the trivial character.
$[\varphi_{4+1}, \chi_1] = 1$, and for $\chi_2 := \varphi_{4+1} - \chi_1$ $[\chi_2, \chi_2] = 1$, so $\chi_2$ is irreducible.
$[\varphi_{3+2}, \chi_1] = 1$, $[\varphi_{3+2}, \chi_2] = 1$, and for $\chi_3 := \varphi_{3+2} - \chi_1 - \chi_2$, $[\chi_3, \chi_3] = 1$, so $\chi_3$ is irreducible.

$[\varphi_{3+1+1}, \chi_1] = 1$, $[\varphi_{3+1+1}, \chi_2] = 2$, $[\varphi_{3+1+1}, \chi_3] = 1$,
and for $\chi_4 := \varphi_{3+1+1} - \chi_1 - 2\chi_2 - \chi_3$, $[\chi_4, \chi_4] = 1$, so $\chi_4$ is irreducible.
Let $\chi_7$ be the alternating character: $\chi_7(g) = 1$ if $g$ is even, and $-1$ if $g$ is odd.
Then $\chi_6 = \chi_2\chi_7$ and $\chi_5 = \chi_3\chi_7$ gives new irreducible characters ($\chi_4\chi_7 = \chi_4$), and this completes the character table.

| | 1 | 15 | 20 | 24 | 10 | 20 | 30 |
|---|---|---|---|---|---|---|---|
| | 1 | (..)(..) | (...) | (.....) | (..) | (...)(..) | (....) |
| $\chi_1$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $\chi_2$ | 4 | 0 | 1 | $-1$ | 2 | $-1$ | 0 |
| $\chi_3$ | 5 | 1 | $-1$ | 0 | 1 | 1 | $-1$ |
| $\chi_4$ | 6 | $-2$ | 0 | 1 | 0 | 0 | 0 |
| $\chi_5$ | 4 | 0 | 1 | $-1$ | $-2$ | 1 | 0 |
| $\chi_6$ | 5 | 1 | $-1$ | 0 | $-1$ | $-1$ | 1 |
| $\chi_7$ | 1 | 1 | 1 | 1 | $-1$ | $-1$ | $-1$ |

(We could have got $\chi_5$, $\chi_6$ and $\chi_7$ in this order if we continued decomposing the permutation characters corresponding to the partitions of type $2+2+1$, $2+1+1+1$ and $1+1+1+1+1$.)

c) Only $\chi_4\big|_{A_5}$ is not irreducible but the other six only gives three different irreducible characters.

| | 1 | 15 | 20 | 12 | 12 |
|---|---|---|---|---|---|
| | 1 | (..)(..) | (...) | $(12345)^{A_5}$ | $(13524)^{A_5}$ |
| $\chi_1$ | 1 | 1 | 1 | 1 | 1 |
| $\chi_2$ | 4 | 0 | 1 | $-1$ | $-1$ |
| $\chi_3$ | 5 | 1 | $-1$ | 0 | 0 |

Let $\omega$ be a primitive 5th root of unity, and let $\varphi$ be the linear character of $\langle a \rangle$ for $a = (12345)$ such that $\varphi(a) = \omega$. Let $\psi = \varphi^{A_5}$. Then $\psi(1) = 12$ and $\psi = 0$ in the second and third columns. The elements $a$ and $a^4$ are conjugate in $A_5$ but $a^2$ and $a^3$ are in the other conjugacy class of $A_5$. So $\psi(g) = 12(\frac{1}{12}\omega + \frac{1}{12}\omega^4) = \omega + \omega^4$, and $\psi(g^2) = \omega^2 + \omega^3$.
$[\psi, \chi_1] = \frac{1}{60}(12 + 12(\omega + \omega^2 + \omega^3 + \omega^4)) = \frac{1}{60}(12 - 12) = 0$,
$[\psi, \chi_2] = \frac{1}{60}(48 - 12(\omega + \omega^2 + \omega^3 + \omega^4)) = 1$,
$[\psi, \chi_3] = \frac{1}{60}(60) = 1$, and for $\chi_4 := \psi - \chi_2 - \chi_3$, $[\chi_4, \chi_4] = 1$, so $\chi_4$ is irreducible, and $\chi_5$ can be calculated by the $2^{nd}$ orthogonality relation. So the complete character

table of $A_5$ is

| | 1<br>$1$ | 15<br>$(..)(..)$ | 20<br>$(...)$ | 12<br>$(12345)^{A_5}$ | 12<br>$(13524)^{A_5}$ |
|---|---|---|---|---|---|
| $\chi_1$ | $1$ | $1$ | $1$ | $1$ | $1$ |
| $\chi_2$ | $4$ | $0$ | $1$ | $-1$ | $-1$ |
| $\chi_3$ | $5$ | $1$ | $-1$ | $0$ | $0$ |
| $\chi_4$ | $3$ | $-1$ | $0$ | $1+\omega+\omega^4$ | $1+\omega^2+\omega^3$ |
| $\chi_5$ | $3$ | $-1$ | $0$ | $1+\omega^2+\omega^3$ | $1+\omega+\omega^4$ |

**HW1.** *Consider the character table of $S_4$.*

| $S_4$ | $1$ | $(..)(..)$ | $(...)$ | $(..)$ | $(....)$ |
|---|---|---|---|---|---|
| $\chi_1$ | $1$ | $1$ | $1$ | $1$ | $1$ |
| $\chi_2$ | $1$ | $1$ | $1$ | $-1$ | $-1$ |
| $\chi_3$ | $2$ | $2$ | $-1$ | $0$ | $0$ |
| $\chi_4$ | $3$ | $-1$ | $0$ | $1$ | $-1$ |
| $\chi_5$ | $3$ | $-1$ | $0$ | $-1$ | $1$ |

*Write the product of the two irreducible characters of degree 3 as a sum of irreducible characters.*

**HW2.** *Determine the character of $S_4$ induced by the trivial character of the 2-Sylow subgroup $\langle(1234),(12)(34)\rangle$, and write it as a sum of irreducible characters.*