1. *Let X be a representation of G and χ the corresponding character. Prove that*
a) $Z(\chi) = \{ g \in G \mid X(g) = \varepsilon I$ *for some* $\varepsilon \in \mathbb{C} \}$*;*
b) $Z(\chi) \triangleleft G$.

Solution: a) $\geq$: If g is in the set on the right and $g^m = 1$, then $X(g)^m = \varepsilon^m I = I$, so $\varepsilon^m = 1$. But then $|\chi(g)| = |\chi(1)\varepsilon| = \chi(1)|\varepsilon| = \chi(1) \Rightarrow g \in Z(g)$.
$\leq$: Suppose that $g \in Z(\chi)$ and the diagonal form of $X(g)$ is $\operatorname{diag}(\varepsilon_1, \ldots, \varepsilon_n)$, where $n = \chi(1)$. Then $\varepsilon_i^m = 1$, so $|\varepsilon_i| = 1$ for every i.

$$\chi(1) = |\chi(g)| = |\varepsilon_1 + \ldots + \varepsilon_n| \leq |\varepsilon_1| + \ldots + |\varepsilon_n| = \chi(1),$$

so the triangle inequality in the middle holds with an equality, thus all ε_i are parallel and point in the same direction in the complex plain. Since their length is also the same, $\varepsilon_1 = \ldots = \varepsilon_n =: \varepsilon$, which gives that $X(g) \sim \varepsilon I$. But every conjugate of a scalar matrix is itself, hence $X(g) = \varepsilon I$.

b) We use the description of $Z(\chi)$ proved in part a). If $g, h \in Z(\chi)$ and $x \in G$, then $X(gh^{-1}) = X(g)X(h^{-1}) = \varepsilon I \overline{\varepsilon'} I = \varepsilon\overline{\varepsilon'} I$ for some $\varepsilon, \varepsilon' \in \mathbb{C}$, and $X(x^{-1}gx) = X(x)^{-1}X(g)X(x) = X(x)^{-1}\varepsilon I X(x) = X(x)^{-1}X(x)\varepsilon I = \varepsilon I$, so $Z(\chi)$ is closed under multiplication, inverse and conjugation.

2. *Prove that the conjugate of any character is a character and the conjugate of an irreducible character is an irreducible character.*

Solution: If X is a representation corresponding to the character χ, then $\overline{X}(g) := \overline{X(g)}$ is also a representation: $\overline{X}(gh) = \overline{X(gh)} = \overline{X(g)X(h)} = \overline{X(g)} \cdot \overline{X(h)} = \overline{X}(g)\overline{X}(h)$, and its character is $\operatorname{tr}(\overline{X}(g)) = \overline{\operatorname{tr}(X(g))} = \overline{\chi(g)}$.
If $[\chi, \chi] = 1$, then $[\overline{\chi}, \overline{\chi}] = \frac{1}{|G|} \sum\limits_{g \in G} \overline{\chi(g)}\chi(g) = \frac{1}{|G|} \sum\limits_{g \in G} \chi(g)\overline{\chi(g)} = [\chi, \chi] = 1$.

3. *a) Prove that* $\sum\limits_{\chi \in \operatorname{Irr} G} \sum\limits_{g \in G} \chi(g)^2$ *is an integer divisible by* $|G|$*.*
b) Prove that the number of self-inverse conjugacy classes is the same as the number of real irreducible characters.

Solution: We can evaluate the sum in two different ways, and it proves both a) and b).

$$\sum_{\chi \in \operatorname{Irr} G} \sum_{g \in G} \chi(g)^2 = \sum_{\chi \in \operatorname{Irr} G} \sum_{g \in G} \chi(g)\overline{\overline{\chi}(g)} = \sum_{\chi \in \operatorname{Irr} G} |G| \cdot [\chi, \overline{\chi}] = \sum_{\substack{\chi \in \operatorname{Irr} G \\ \chi = \overline{\chi}}} |G|$$

is $|G|$ times the number of real irreducible characters (so clearly an integer multiple of $|G|$).

$$\sum_{\chi \in \operatorname{Irr} G} \sum_{g \in G} \chi(g)^2 = \sum_{g \in G} \sum_{\chi \in \operatorname{Irr} G} \chi(g)^2 = \sum_{g \in G} \sum_{\chi \in \operatorname{Irr} G} \chi(g)\overline{\chi(g^{-1})}.$$

By the second orthogonality relation, the inside sum is 0 if $g \not\sim g^{-1}$, and $|C_G(g)|$ if $g \sim g^{-1}$. So the total sum is further equal to

$$\sum_{g \sim g^{-1}} |C_G(g)| = \sum_{\substack{\mathcal{K}_i \text{ conj.cl.} \\ \mathcal{K}_i = \mathcal{K}_i^{-1}}} |\mathcal{K}_i||C_G(g_i)| = \sum_{\substack{\mathcal{K}_i \text{ conj.cl.} \\ \mathcal{K}_i = \mathcal{K}_i^{-1}}} |G|$$

which is $|G|$ times the number of self-inverse conjugacy classes. Comparing the two results we get the statement of part b).

4. *Let χ be a character of the group G. Consider the map $\det\chi : G \to \mathbb{C}^\times$, where $(\det\chi)(g) = \det X(g)$ and X is a representation for the character χ. Prove that $\det\chi$ is a well-defined linear character.*

Solution: If X and X' are both representations of for the same character χ then X and X' are equivalent, i.e. there exists an invertible matrix P such that $P^{-1}X(g)P = X'(g)$ for every $g \in G$. This implies that $\det X'(g) = \det(P^{-1}X(g)P) = \det X(g)$, so $\det\chi$ is well-defined. Also, $(\det\chi)(gh) = \det X(gh) = \det X(g)\cdot\det X(h) = (\det\chi)(g)\cdot(\det\chi)(h)$, so $\det\chi$ is a group homomorphism from G to $\mathbb{C}^\times$, in other words, $\det\chi$ is a linear character.

5. *Prove that a simple group cannot have an irreducible character of degree 2. (Use Problem 5.)*

Solution: Suppose that G is simple and $\chi \in \operatorname{Irr} G$ has degree 2. Then G cannot be abelian, and $\chi \neq 1_G$. Since $\operatorname{Ker}\chi$ is a normal subgroup, and it is not the whole G, we get that $\operatorname{Ker}\chi = 1$. Furthermore, $Z(\chi) = Z(\chi)/\operatorname{Ker}\chi = Z(G/\operatorname{Ker}\chi) = Z(G) \neq G$ gives that $Z(\chi) = 1$. We also know that $\chi(1) \mid |G|$ for any irreducible χ, thus $|G|$ is even, which implies that G has an element g of order 2. For this g, $X(g)^2 = I$, so $X(g) \sim \begin{bmatrix} \varepsilon_1 & 0 \\ 0 & \varepsilon_2 \end{bmatrix}$, where $\varepsilon_i = \pm 1$. If $\varepsilon_1 \neq \varepsilon_2$, then $\det X(g) \neq 1$, so $\det\chi$ would be a nontrivial linear character, so $G' < G$, which cannot happen for a nonabelian simple group. So $\varepsilon_1 = \varepsilon_2$, thus $g \in Z(\chi) = Z(G)$, which is again a contradiction.

A conjugacy class of even permutations of S_n is either also a conjucacy class of A_n or it splits into two conjugacy classes of equal size. The conjugacy class splits if and only if the cyclic decomposition contains only cycles of odd length and it contains at most one cycle of any length (including the 1-cylces).

6. *a) List the conjugacy classes of S_6 and determine their sizes.*
b) List the conjugacy classes of A_8 and determine the sizes of those that are not complete conjugacy classes in S_8.

Solution:

a) The conjugacy classes of S_n consist of elements of the same type of cyclic decomposition, so we only have to list the possible partitions of n. Write the cycle lengths in decreasing order, and write the decomposition types in increasing lexicographic order.

	size of conj. class
$1+1+1+1+1+1$,	1
$2+1+1+1+1$,	$\binom{6}{2} = 15$
$2+2+1+1$,	$\binom{6}{2}\binom{4}{2}\frac{1}{2!} = 45$
$2+2+2$,	$\binom{6}{2}\binom{4}{2}\frac{1}{3!} = 15$
$3+1+1+1$,	$\binom{6}{3}\cdot 2! = 40$
$3+2+1$,	$\binom{6}{3}\binom{3}{2}2! = 120$
$3+3$,	$\binom{6}{3}\frac{1}{2!}(2!)^2 = 40$
$4+1+1$,	$\binom{6}{4}\cdot 3! = 90$
$4+2$,	$\binom{6}{4}3! = 90$
$5+1$,	$\binom{6}{5}4! = 144$
6	$\binom{6}{6}5! = 120$

b) Here we only have to list those partitions of 8, where the number of even summands is even.

$1+1+1+1+1+1+1+1$
$2+2+1+1+1+1$
$2+2+2+2$
$3+1+1+1+1+1$
$3+2+2+1$
$3+3+1+1$
$4+2+1+1$
$4+4$
$5+1+1+1$
$5+3$
$6+2$
$7+1$

Here only $5+3$ and $7+1$ satisfy the condition for a conjugacy class of S_8 to split in A_8. The size of the first is $\binom{8}{5}\cdot 4!\cdot 2! = 2688$, the size of the second is $\binom{8}{7}\cdot 6! = 8\cdot 6! = 5760$. So the sizes of the correscponding conjugacy classes of A_8 are 1344 and 2880.

7. *a) Consider the dihedral group D_4 to be a subgroup of S_4 by the action on a square with corners $1,2,3,4$. Determine the permutation character of D_4 defined by its action on the two-element subsets of $\{1,2,3,4\}$.*

b) Determine the permutation character of D_4 defined by the action on all 3-colourings of the corners of the square.

c) Determine the permutation character of S_6 defined by the action on the partitions of type $3+3$.

d) Determine the scalar product of the characters in a), b) and c) with the trivial character (that is, the number of orbits of the given group action), and the scalar square of the character. What can we say about the irreducible components?

Solution: a) The two-element subsets can be considered as the four sides and the two diagonals of the square. The identity leaves all six of them fixed. The rotation by 180° leaves the diagonals fixed but the sides are swapped so the number of the fixed points is 2. The rotations by 90° have no fixed elements among the sides and diagonals. The reflection across a diagonal fixes only the diagonals. Finally, a reflection across an axis parallel to two of the sides of the square leaves the other two sides fixed. So if we denote the rotation by 90° by f, and the reflection across a diagonal by t, then the character is

	1	1	2	2	2
	1	f^2	$f^{\pm 1}$	t, tf^2	tf, tf^3
χ	6	2	0	2	2

b) There are $3^4 = 81$ colourings, and 1 fixes them all. f^2 fixes those where the corners on the same diagonal are coloured the same way. This gives $3^2 = 9$ possibilities. f and f^{-1} fix only those colourings, where all corners have the same colour: this means 3 possibilities. The reflection across a diagonal fixes those where the colors of the corners not on that diagonal are the same, i.e. $3^3 = 27$ colourings. Finally the other type of reflection fixes those where the colour of any corner is the same as its mirror image on the other side, so there are only $3^2 = 9$ possibilities. This gives the following permutation character.

	1	1	2	2	2
	1	f^2	$f^{\pm 1}$	t, tf^2	tf, tf^3
ψ	81	9	3	27	9

c) There are altogether $\binom{6}{3} = 20$ such partitions. (12)(34) fixes those where $1, 2$ go into one class, $3, 4$ into the other, and 5 goes to any of the two, 6 to the other. This is $2 \cdot 2 = 4$ possibilities. (123) fixes those where $1, 2, 3$ go into one of the two classes, and $4, 5, 6$ to the other. This gives two possibilities. If we ontinue counting the fixed partions, we get the following character.

	1	45	40	40	90	144	15	15	120	90	120
	1	(..)(..)	(...)	(...)(...)	(....)(..)	(.....)	(..)	(..)(..)(..)	(...)(..)	(....)	(......)
φ	20	4	2	2	0	0	8	0	2	0	0

d) From part a), $[\chi, 1_{D_4}] = \frac{1}{8}(6 + 2 + 2 \cdot 2 + 2 \cdot 2) = 2$, and, indeed, this permutation has two orbits, the sides and the diagonals.
$[\chi, \chi] = \frac{1}{8}(36 + 4 + 4 \cdot 2 + 4 \cdot 2) = 7$. So the sum of the squares of the coefficients of the irreducible characters in the decomposition of χ is 7. But one coefficient is 2 (this is the coefficient of 1_{D_4}), so $7 = 2^2 + 1^1 + 1^2 + 1^2$ is the only possible decomposition: three other irreducible characters appear with coefficient 1. (Actually, from $\chi(1) = 6$ it also follows that the second degree character is also a component.) We could determine the exact decomposition if we calculated the scalar product with all the irreducible characters.
From part b), $[\psi, 1_{D_4}] = \frac{1}{8}(81 + 9 + 3 \cdot 2 + 27 \cdot 2 + 9 \cdot 2) = 21$, so there are 21 different 3-colourings of the corners of the square, up to isometries.
$[\psi, \psi] = \frac{1}{8}(6561 + 81 + 9 \cdot 2 + 729 \cdot 2 + 81 \cdot 2) = 1035$. Here we can say even less about the decomposition of the character without calculating the scalar products.

	1	1	2	2	2
	1	f^2	$f^{\pm 1}$	t, tf^2	tf, tf^3
χ_1	1	1	1	1	1
χ_2	1	1	-1	-1	1
χ_3	1	1	-1	1	-1
χ_4	1	1	1	-1	-1
χ_5	2	-2	0	0	0

Using this character table, we get that $\chi = 2\chi_1 + \chi_2 + \chi_3 + \chi_5$ and $\psi = 21\chi_1 + 6\chi_2 + 15\chi_3 + 3\chi_4 + 18\chi_5$.
From part c), $[\varphi, 1_{S_6}] = \frac{1}{720}(20 + 4 \cdot 45 + 2 \cdot 40 + 2 \cdot 40 + 8 \cdot 15 + 2 \cdot 120) = 1$, and $[\varphi, \varphi] = \frac{1}{720}(400 + 16 \cdot 45 + 4 \cdot 40 + 4 \cdot 40 + 64 \cdot 15 + 4 \cdot 120) = 4$. Since 1_{S_6} appears only once in the decomposition, φ can only be the sum of four different irreducible characters.

HW1. *Determine the value of* $\sum_{\chi_i, \chi_j \in \mathrm{Irr}\, G} \sum_{g \in G} \chi_i(1)\chi_j(1)\chi_i(g)\chi_j(g)$.

HW2. *Determine the permutation character* χ *of* S_4 *defined by the action on the partitions of type* $4 + 2$. *Calculate* $[\chi, \chi]$. *How many irreducible components can* χ *have?*