Standard Koszul standardly stratified algebras

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Abstract

In this paper, we prove that every standard Koszul (not necessarily graded) standardly stratified algebra is also Koszul. This generalizes a similar result of [3] on quasi-hereditary algebras.

Keywords: *standardly stratified algebras, standard Koszul algebras, Koszul algebras*
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Ágoston, Dlab and Lukács [3] gave a sufficient, and in the graded case, also necessary condition for a quasi-hereditary algebra to have a quasi-hereditary Yoneda extension algebra. They called these algebras standard Koszul, meaning that their right and left standard modules are Koszul modules, i.e. the right and left standard modules have top projective resolutions. As a part of the proof, they also showed that if a quasi-hereditary algebra is standard Koszul then its simple modules are also Koszul modules, in other words, the algebra itself is a Koszul algebra.

In [4], they generalized their earlier result to (graded) standardly stratified algebras. Unlike in the quasi-hereditary case, where the filtration of the algebra by right standard modules corresponds to the filtration by left standard modules, here the left regular module is filtered by proper standard modules. Naturally, the concept of standard Koszul algebras also had to be modified: the right standard and the left proper standard modules should be Koszul (in the quasi-hereditary case, the standard and proper standard modules coincide). They proved that a graded standard Koszul standardly stratified Koszul algebra has a standardly stratified extension algebra. However, the question whether the standard Koszul property here also implies that the algebra is Koszul, remained open. We should point out here that for quasi-hereditary algebras, this implication was useful in several situations (cf. [5], [8], [10] or [11]).

In this paper, we settle the question even in the more general, non-graded setting, proving the following theorem.

Theorem. *If A is a standard Koszul standardly stratified algebra, then A is a Koszul algebra.*

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Preliminaries

Throughout the paper, A is a basic finite dimensional K-algebra, where K is an arbitrary field. The Jacobson radical of A will be denoted by J, and we write $\hat{S} = A/J$. All A-modules are meant to be right A-modules unless otherwise stated. Let us fix a complete ordered set of primitive orthogonal idempotents $e_1, \dots, e_n$ in A. The ith indecomposable projective component e_iA of the regular module $A_A = e_1A \oplus \dots \oplus e_nA$ is denoted by $P(i)$ and its simple top $P(i)/\operatorname{rad} P(i)$ by $S(i)$. The corresponding left modules are denoted by $P^\circ(i)$ and $S^\circ(i)$, respectively. Let ε_i stand for the idempotent $e_i + e_{i+1} + \dots + e_n$, and set $\varepsilon_{n+1} = 0$. We write C_i for the ith centralizer algebra $\varepsilon_i A \varepsilon_i$, where the ordering of the idempotents $e_i, \dots, e_n$ is inherited from A.

Having an algebra A with an ordered set of primitive orthogonal idempotents, one can define the standard and proper standard A-modules as follows. The ith standard module is $\Delta(i) = e_iA/e_iA\varepsilon_{i+1}A$, while the ith proper standard module is $\overline{\Delta}(i) = e_iA/e_iJ\varepsilon_iA$. That is, the ith standard module is the largest factor module of $P(i)$ which has no composition factor $S(j)$ with $j > i$, and the ith proper standard module is the largest factor module of $P(i)$ whose radical has no composition factor $S(j)$ with $j \geq i$. The left standard and proper standard modules $\Delta^\circ(i)$ and $\overline{\Delta}^\circ(i)$ can be defined analogously. The algebra A is said to be standardly stratified, if Ae_nA is a projective module and the factor algebra A/Ae_nA is again standardly stratified, or in other words, if the regular module A_A is filtered by the modules $\Delta(i)$. By [7], this is equivalent to the condition that ${}_AA$ is filtered by the proper standard modules $\overline{\Delta}^\circ(i)$.

We say that a submodule X of Y is a *top submodule*, and write $X \overset{t}{\leq} Y$, whenever $X \cap \operatorname{rad} Y = \operatorname{rad} X$, i.e. the natural embedding of X into Y induces an embedding of $X/\operatorname{rad} X$ into $Y/\operatorname{rad} Y$. The class $\mathcal{C}_A^i \subseteq$ mod-A consists of those modules X whose minimal projective resolution

$$\cdots \to P_k \to \cdots \to P_1 \to P_0 \to X \to 0$$

is a *top resolution* up to the ith term, that is, the kth syzygy, Ω_k is a top submodule of $\operatorname{rad} P_{k-1}$ for all $k \leq i$. We call a module X a Koszul module if $X \in \mathcal{C}_A = \cap_{i=1}^{\infty} \mathcal{C}_A^i$. The algebra A is standard Koszul if $\Delta(i) \in \mathcal{C}_A$ and $\overline{\Delta}^\circ(i) \in \mathcal{C}_{A^\circ}$ for all i, while A is a Koszul algebra if every simple module $S(i)$ is a Koszul module. (Note that the latter condition implies that the left simple modules are also Koszul, see for example [9]).

We should mention here that the Koszul modules and Koszul algebras defined here are traditionally called quasi-Koszul, but here we use the simpler term, in accordance with the terminology in [3], since this concept is the natural extension of the notion of Koszul modules over graded algebras to the non-graded setting.

1 Lean algebras

Among quasi-hereditary algebras, lean algebras are those which satisfy the condition $e_iJ^2e_j = e_iJ\varepsilon_mJe_j$ for every i, j with $m = \min\{i, j\}$. It was shown in [1] that this condition is equivalent to saying that $\Delta(i) \in \mathcal{C}_A^1$ and $\Delta(j)^\circ \in \mathcal{C}_{A^\circ}^1$ for every i, j. Closely following the proof, we get the next statement about the analogue of lean algebras in the standardly stratified setting.

Lemma 1.1. *The algebra A satisfies the condition that $\Delta(i) \in \mathcal{C}_A^1$ and $\overline{\Delta}(j)^\circ \in \mathcal{C}_{A^\circ}^1$ for every i, j if and only if $e_iJ^2e_j = e_iJ\varepsilon_mJe_j$ for every i, j with $m = \min\{i+1, j\}$.*

Proof. $\Delta(i) \in \mathcal{C}_A^1 \iff e_iA\varepsilon_{i+1}A = e_iJ\varepsilon_{i+1}A \overset{t}{\leq} e_iJ \iff e_iJ\varepsilon_{i+1}A \cap e_iJ^2 \subseteq e_iJ\varepsilon_{i+1}J \iff e_iJ\varepsilon_{i+1}Ae_j \cap e_iJ^2e_j \subseteq e_iJ\varepsilon_{i+1}Je_j\ \forall j$. For $j \leq i$, $e_iJ\varepsilon_{i+1}Ae_j = e_iJ\varepsilon_{i+1}Je_j$, so the last inclusion is always true for $j \leq i$. On the other hand, for $j \geq i+1$, we have $e_iJ\varepsilon_{i+1}Ae_j = e_iJe_j \supseteq e_iJ^2e_j$, so

$$\Delta(i) \in \mathcal{C}_A^1 \iff e_iJ^2e_j \subseteq e_iJ\varepsilon_{i+1}Je_j\ \forall j.$$

$\overline{\Delta}^\circ(j) \in \mathcal{C}_{A^\circ}^1 \iff A\varepsilon_jJe_j \overset{t}{\leq} Je_j \iff A\varepsilon_jJe_j \cap J^2e_j \subseteq J\varepsilon_jJe_j \iff e_iA\varepsilon_jJe_j \cap e_iJ^2e_j \subseteq e_iJ\varepsilon_jJe_j\ \forall i$. For $i < j$, $e_iA\varepsilon_jJe_j = e_iJ\varepsilon_jJe_j$, so the last inclusion is always true for $i < j$. On the other hand, for $i \geq j$, we have $e_iA\varepsilon_jJe_j = e_iJe_j \supseteq e_iJ^2e_j$, so

$$\overline{\Delta}^\circ(j) \in \mathcal{C}_{A^\circ}^1 \iff e_iJ^2e_j \subseteq e_iJ\varepsilon_jJe_j\ \forall i.$$

The combination of the two conditions (and the trivial reverse inclusion) gives the statement of the lemma. □

In particular, standard Koszul algebras satisfy the condition of the previous lemma. As a consequence, we get a useful feature of standard Koszul algebras in terms of the idempotents ε_i.

Corollary 1.2. *If $\Delta(i) \in \mathcal{C}_A^1$ and $\overline{\Delta}(j)^\circ \in \mathcal{C}_{A^\circ}^1$ for every i, j (in particular, if A is standard Koszul), then $\varepsilon_iJ^2\varepsilon_i = \varepsilon_iJ\varepsilon_iJ\varepsilon_i$ for every i.*

The next few lemmas will be useful in finding connection between top embeddings over A and those over its centralizer algebras (cf. [1]).

Lemma 1.3. *If $X \leq Y \leq Z$, and $X \overset{t}{\leq} Z$, then*

(1) $X \overset{t}{\leq} Y$;

(2) $Y \overset{t}{\leq} Z \Leftrightarrow Y/X \overset{t}{\leq} Z/X$.

Lemma 1.4. *Let ε be an idempotent in A, and $X \leq Y$ be A-modules such that $X = X\varepsilon A$ and $Y = Y\varepsilon A$. Then*
(1) $X \overset{t}{\leq} Y \Leftrightarrow X\varepsilon \overset{t}{\leq} Y\varepsilon$ in mod-$\varepsilon A\varepsilon$.

(2) If we also assume that $\varepsilon J^2\varepsilon = \varepsilon J\varepsilon J\varepsilon$, then $X \overset{t}{\leq} \operatorname{rad} Y \Leftrightarrow X\varepsilon \overset{t}{\leq} \operatorname{rad} Y\varepsilon$ in mod-$\varepsilon A\varepsilon$.

Proof. If $X \cap YJ \subseteq XJ$, then $X\varepsilon \cap Y\varepsilon J\varepsilon = (X \cap YJ)\varepsilon = XJ\varepsilon = \operatorname{rad} X\varepsilon$. Conversely, if $X\varepsilon \cap YJ\varepsilon = X\varepsilon J\varepsilon$, then $(X \cap YJ)\varepsilon = X\varepsilon \cap YJ\varepsilon = X\varepsilon J\varepsilon \subseteq XJ$, while $(X \cap YJ)(1-\varepsilon) \subseteq X\varepsilon A(1-\varepsilon) \subseteq XJ$, so $X \cap YJ \subseteq XJ$.

The second statement is contained in Lemma 1.6 of [3]. □

Lemma 1.5. *Let $\varepsilon \in A$ be an idempotent element. Suppose that $X \le Y$ are two A-modules such that $X\varepsilon A = 0$. Then*

$$(Y/X)\varepsilon A \overset{t}{\le} Y/X \Leftrightarrow Y\varepsilon A \overset{t}{\le} Y.$$

Proof. Rewrite the condition $Y\varepsilon A \overset{t}{\le} Y$ as $Y\varepsilon A \cap YJ \subseteq Y\varepsilon J$ and the condition $(Y/X)\varepsilon A \overset{t}{\le} Y/X$ as $Y\varepsilon A \cap (YJ + X) \subseteq Y\varepsilon J + X$. Since $Y\varepsilon A(1-\varepsilon) \subseteq Y\varepsilon J$ and $X\varepsilon = 0$, both of the previous inclusions are equivalent to $Y\varepsilon \cap YJ\varepsilon \subseteq Y\varepsilon J\varepsilon$. □

Lemma 1.6. *Let $0 \to X \to Y \to Z \to 0$ be a short exact sequence with $X\varepsilon A \overset{t}{\le} Y$. Then $Y\varepsilon A \overset{t}{\le} Y$ if and only if $Z\varepsilon A \overset{t}{\le} Z$.*

Proof. Take the factors of X and Y by $X\varepsilon A$ to get

$$0 \to \overline{X} \to \overline{Y} \to Z \to 0.$$

By Lemma 1.3, $Y\varepsilon A \overset{t}{\le} Y$ if and only if $\overline{Y}\varepsilon A \overset{t}{\le} \overline{Y}$. Since $\overline{X}\varepsilon A = 0$, the latter is equivalent to $Z\varepsilon A \overset{t}{\le} Z$ by Lemma 1.5. □

We shall need a generalized version of Lemma 1.6.

Lemma 1.7. *Let ε be an idempotent in A. Suppose that the following commutative diagram has exact rows and columns.*

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & X_1 & \longrightarrow & Y_1 & \xrightarrow{\alpha_1} & Z_1 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & X & \longrightarrow & Y & \xrightarrow{\alpha} & Z & \longrightarrow & 0 \end{array}$$

If $X_1\varepsilon A \overset{t}{\le} Y$ and $Z_1\varepsilon A \overset{t}{\le} Z$, then $Y_1\varepsilon A \overset{t}{\le} Y$.

Proof. We may assume that $X_1\varepsilon A = 0$ because otherwise we can substitute the modules X_1, X, Y_1 and Y with their factors by the (top) submodule $X_1\varepsilon A$. In the new diagram, the same embeddings will be top embeddings as in the original by Lemma 1.3. Then

$$X_1 \cap Y_1\varepsilon A = X_1(1-\varepsilon) \cap Y_1\varepsilon A \subseteq Y_1\varepsilon A(1-\varepsilon) \subseteq Y_1\varepsilon J.$$

The assumption $Z_1\varepsilon A \cap ZJ \subseteq Z_1\varepsilon J$ implies that

$$(Y_1\varepsilon A \cap YJ)\alpha_1 \subseteq (Y_1\varepsilon A)\alpha_1 \cap (YJ)\alpha = Z_1\varepsilon A \cap ZJ \subseteq Z_1\varepsilon J = (Y_1\varepsilon J)\alpha_1,$$

so $Y_1\varepsilon A \cap YJ \subseteq Y_1\varepsilon J + X_1$, thus

$$Y_1\varepsilon A \cap YJ \subseteq Y_1\varepsilon A \cap (X_1 + Y_1\varepsilon J) = (Y_1\varepsilon A \cap X_1) + Y_1\varepsilon J \subseteq Y_1\varepsilon J,$$

giving that $Y_1\varepsilon A \overset{t}{\leq} Y$. □

Remark 1.8. Note that the "reverse" of Lemma 1.7 does not hold in general. Let $X \leq Y$ and suppose that X is not a top submodule of Y. Consider the following commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & 0 & \longrightarrow & X & \longrightarrow & X & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \alpha} & & \\
0 & \longrightarrow & X & \longrightarrow & X \oplus Y & \longrightarrow & Y & \longrightarrow & 0
\end{array}$$

with exact rows and columns, where β is the diagonal map and the bottom row splits. Here, β is a top embedding but α is not.

Finally, we would like to recall Lemma 1.7 from [3] about the connection between Koszul A- and $\varepsilon A\varepsilon$-modules.

Lemma 1.9. *Suppose that ε is an idempotent of A such that $\varepsilon J^2\varepsilon = \varepsilon J\varepsilon J\varepsilon$, and let X be a module with $\operatorname{Ext}_A^t(X, \operatorname{top}((1-\varepsilon)A)) = 0$ for all $t \geq 0$. Then $X \in \mathcal{C}_A$ if and only if $X\varepsilon \in \mathcal{C}_{\varepsilon A\varepsilon}$.*

2 Standard Koszul standardly stratified algebras

In this section, we turn our attention to standardly stratified algebras. Before we prove our main theorem, we present some preparatory lemmas. These lemmas lay the foundation of an inductive method which involves the centralizer algebras of a standardly stratified algebra.

Lemma 2.1. *Suppose that A is a standard Koszul standardly stratified algebra. Then its centralizer algebra $C_2 = \varepsilon_2 A\varepsilon_2$ is again standard Koszul and standardly stratified, its standard and left proper standard modules are $\Delta(i)\varepsilon_2$ and $\varepsilon_2\overline{\Delta}^{\circ}(i)$ for $i \geq 2$.*

Proof. Observe that $(\varepsilon_2 A\varepsilon_2)e_n(\varepsilon_2 A\varepsilon_2) = \varepsilon_2(Ae_nA)\varepsilon_2$ is a projective C_2-module, since Ae_nA is the direct sum of copies of e_nA, and $\varepsilon_2 e_n A\varepsilon_2 = e_nC_2$. So C_2 is standardly stratified because $\varepsilon_2 A\varepsilon_2/\varepsilon_2 Ae_nA\varepsilon_2 \cong \varepsilon_2(A/Ae_nA)\varepsilon_2$ as algebras. It is also easy to check that the standard modules $\Delta_{C_2}(i)$ and the left proper standard modules $\overline{\Delta}^{\circ}_{C_2}(i)$ $(i \geq 2)$ over C_2 are isomorphic to the modules $\Delta(i)\varepsilon_2$ and $\varepsilon_2\overline{\Delta}^{\circ}(i)$, respectively. The Koszul property of the modules $\Delta(i)\varepsilon_2$ and $\varepsilon_2\overline{\Delta}^{\circ}(i)$ follows from Lemma 1.9, since $\operatorname{Ext}_A^t(\Delta(i), S(1)) = 0 = \operatorname{Ext}_A^t(\overline{\Delta}^{\circ}(i), S^{\circ}(1))$ for any $t \geq 0$ and $i \geq 2$. □

Let S be a semisimple A-module. As in Definition 1.8 of [3], a module X is called S-Koszul, if $\mathrm{Ext}_A^t(X,S) \subseteq \mathrm{Ext}_A^1(\hat{S},S)\cdot\mathrm{Ext}_A^{t-1}(X,\hat{S})$ for all $t \geq 1$, or equivalently, the trace of S in the top of the syzygy $\Omega_t(X)$ is mapped injectively into the top of rad $P_{t-1}(X)$ for every $t \geq 1$. In other words, X is S-Koszul if and only if $\Omega_t(X)\varepsilon_S A \cap P_{t-1}(X)J^2 \subseteq \Omega_t(X)J$ for every $t \geq 1$, where $\varepsilon_S = \sum\{e_i \mid Se_i \neq 0\}$.

Lemma 2.2. *A module X is Koszul if and only if X is $S(1)$-Koszul and $\Omega_t(X)\varepsilon_2 A \cap P_{t-1}(X)J^2 \subseteq \Omega_t(X)J$ for all $t \geq 1$, i. e. X is both $S(1)$- and $\oplus_{i\geq 2}S(i)$-Koszul.*

Proof. For $X \leq Y$, the condition $X \cap YJ \subseteq XJ$ holds if and only if $Xe_1A \cap YJ \subseteq XJ$ and $X\varepsilon_2 A \cap YJ \subseteq XJ$. □

Corollary 2.3. *If X is an $S(1)$-Koszul module and $\Omega_t(X)\varepsilon_2 A \overset{t}{\leq} \mathrm{rad}\ P_{t-1}(X)$ for all $t \geq 0$, then $X \in \mathcal{C}_A$.*

Now let us take the subclass $\mathcal{K}$ of A-modules

$$\mathcal{K} = \left\{ X \mid X \text{ is } S(1)\text{-Koszul},\ X\varepsilon_2 A \overset{t}{\leq} X,\ X\varepsilon_2 \in \mathcal{C}_{C_2} \right\}.$$

As in the case of quasi-hereditary algebras in [3], we plan to show that all modules in $\mathcal{K}$ are Koszul, and the simple modules belong to $\mathcal{K}$. First we investigate modules without the additional $S(1)$-Koszul property:

$$\mathcal{K}_2 = \left\{ X \mid X\varepsilon_2 A \overset{t}{\leq} X,\ X\varepsilon_2 \in \mathcal{C}_{C_2} \right\}.$$

We fix some notation for the upcoming lemmas. For any A-module X, let $\mathcal{P}(X)$ and $\Omega(X)$ denote the projective cover and the first syzygy of X, respectively, while $\tilde{X}$ will stand for the submodule $X\varepsilon_2 A$, and $\overline{X}$ for the respective factor module $X/X\varepsilon_2 A$.

For the rest of the section, A is always assumed to be a standard Koszul standardly stratified algebra.

Lemma 2.4. *If X is an A-module for which $X\varepsilon_2 A = 0$, then $\Omega(X)\varepsilon_2 A$ is a top submodule of* rad $\mathcal{P}(X)$.

Proof. Since $X\varepsilon_2 A = 0$, the projective cover $\mathcal{P}(X)$ is isomorphic to $\oplus P(1)$, and $\Omega(X)\varepsilon_2 A = (\mathrm{rad}\ \oplus P(1))\varepsilon_2 A = \oplus P(1)\varepsilon_2 A \overset{t}{\leq} \mathrm{rad}\ \oplus P(1)$ as $\Delta(1) \in \mathcal{C}_A^1$. □

Lemma 2.5. *Let X be an arbitrary A-module. If $X \in \mathcal{K}_2$, then $\Omega(\tilde{X}) \in \mathcal{K}_2$. Moreover, $\Omega(\tilde{X})\varepsilon_2 A \overset{t}{\leq} \mathrm{rad}\, P(\tilde{X})$.*

Proof. Take the minimal projective resolution $0 \to \Omega(\tilde{X}) \to \mathcal{P}(\tilde{X}) \to \tilde{X} \to 0$ of $\tilde{X}$, and apply the exact functor $\mathrm{Hom}_A(\varepsilon_2 A, -)$ to get the short exact sequence $0 \to \Omega(\tilde{X})\varepsilon_2 \to \mathcal{P}(\tilde{X})\varepsilon_2 \to \tilde{X}\varepsilon_2 \to 0$ of C_2-modules. Since $\tilde{X} = \tilde{X}\varepsilon_2 A$, the projective module $\mathcal{P}(\tilde{X})\varepsilon_2$ is the projective cover of $\tilde{X}\varepsilon_2$. But $X \in \mathcal{K}_2$ gives that $\tilde{X}\varepsilon_2 = X\varepsilon_2$ belongs to $\mathcal{C}_{C_2}$, together with its syzygy $\Omega(\tilde{X})\varepsilon_2$.

Furthermore, we have $\Omega(\tilde{X})\varepsilon_2 \overset{t}{\leq} \mathrm{rad}\ \mathcal{P}(\tilde{X})\varepsilon_2$, so by Lemma 1.4, $\Omega(\tilde{X})\varepsilon_2 A \overset{t}{\leq} \mathrm{rad}\ \mathcal{P}(\tilde{X})$, and also $\Omega(\tilde{X})\varepsilon_2 A \overset{t}{\leq} \Omega(\tilde{X})$ by Lemma 1.3. □

Proposition 2.6. *The class $\mathcal{K}_2$ is closed under syzygies, that is if $X \in \mathcal{K}_2$, then $\Omega(X)\varepsilon_2 A \overset{t}{\leq} \Omega(X)$ and $\Omega(X)\varepsilon_2 \in \mathcal{C}_{C_2}$.*

Proof. Consider the commutative diagram

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \Omega(\tilde{X}) & \longrightarrow & \Omega(X) & \longrightarrow & \Omega(\overline{X}) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{P}(\tilde{X}) & \longrightarrow & \mathcal{P}(X) & \longrightarrow & \mathcal{P}(\overline{X}) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \tilde{X} & \longrightarrow & X & \longrightarrow & \overline{X} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array} \qquad (1)$$

The condition $\tilde{X} \overset{t}{\leq} X$ implies that $\mathrm{top}\ X \cong \mathrm{top}\ \tilde{X} \oplus \mathrm{top}\ \overline{X}$, so the projective module $\mathcal{P}(X)$ in the middle of the diagram is indeed the projective cover of X.

By Lemma 2.5, $\Omega(\tilde{X})\varepsilon_2 A \overset{t}{\leq} \mathrm{rad}\ \mathcal{P}(\tilde{X})$ and $\Omega(\tilde{X})\varepsilon_2 A \overset{t}{\leq} \Omega(\tilde{X})$. The former implies $\Omega(\tilde{X})\varepsilon_2 A \overset{t}{\leq} \mathrm{rad}\ \mathcal{P}(X)$ because the middle row splits, so we also have $\Omega(\tilde{X})\varepsilon_2 A \overset{t}{\leq} \Omega(X)$ by Lemma 1.3. We can apply Lemma 1.6 to the first row of the diagram to get $\Omega(X)\varepsilon_2 A \overset{t}{\leq} \Omega(X)$, as $\Omega(\overline{X})\varepsilon_2 A \overset{t}{\leq} \Omega(\overline{X})$ holds by Lemma 2.4 and Lemma 1.3. The first statement of the lemma is now proved.

Let us apply the functor $\mathrm{Hom}_A(\varepsilon_2 A, -)$ to the first row and the third column of diagram (1).

$$\begin{array}{ccccccccc} & & & & & & 0 & & \\ & & & & & & \downarrow & & \\ 0 & \longrightarrow & \Omega(\tilde{X})\varepsilon_2 & \longrightarrow & \Omega(X)\varepsilon_2 & \longrightarrow & \Omega(\overline{X})\varepsilon_2 & \longrightarrow & 0 \\ & & & & & & \downarrow & & \\ & & & & & & \mathcal{P}(\overline{X})\varepsilon_2 & & \\ & & & & & & \downarrow & & \\ & & & & & & \overline{X}\varepsilon_2 & & \\ & & & & & & \downarrow & & \\ & & & & & & 0 & & \end{array}$$

It is clear that both the row and the column are exact. As $\overline{X}\varepsilon_2 = 0$, the modules $\Omega(\overline{X})\varepsilon_2$ and $\mathcal{P}(\overline{X})\varepsilon_2$ are isomorphic, and the latter can be written in the form $\oplus P(1)\varepsilon_2$. The module $\oplus P(1)\varepsilon_2$ is Koszul because $\Delta(1)$, and also its syzygy, $P(1)\varepsilon_2 A$ are Koszul modules satisfying the conditions of Lemma 1.9. So $\Omega(\overline{X})\varepsilon_2$ is a Koszul C_2-module.

Let us observe also that $\Omega(\tilde{X})\varepsilon_2$ is Koszul by Lemma 2.5, so the first and the last terms of the exact sequence

$$0 \to \Omega(\tilde{X})\varepsilon_2 \to \Omega(X)\varepsilon_2 \to \Omega(\overline{X})\varepsilon_2 \to 0$$

are Koszul. Besides, we have seen that $\Omega(\tilde{X})\varepsilon_2 A \overset{t}{\leq} \Omega(X)$, thus Lemmas 1.3 and 1.4 give that the map $\Omega(\tilde{X})\varepsilon_2 \to \Omega(X)\varepsilon_2$ is a top embedding. So by Lemma 2.4 of [2], the C_2-module $\Omega(X)\varepsilon_2$ is also Koszul. □

Proposition 2.7. *All modules in $\mathcal{K}_2$ are $\oplus_{i\geq 2}S(i)$-Koszul.*

Proof. In view of the previous proposition and Corollary 2.3, it suffices to prove that $X \in \mathcal{K}_2$ implies $\Omega(X)\varepsilon_2 A \overset{t}{\leq} \operatorname{rad} \mathcal{P}(X)$, and the rest will follow by induction. Let $X \in \mathcal{K}_2$, and take a look at diagram (1) again. Since – as we noted – its middle row is split exact, we have the commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Omega(\tilde{X}) & \longrightarrow & \Omega(X) & \longrightarrow & \Omega(\overline{X}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \operatorname{rad} \mathcal{P}(\tilde{X}) & \to & \operatorname{rad} \mathcal{P}(X) & \longrightarrow & \operatorname{rad} \mathcal{P}(\overline{X}) & \longrightarrow & 0
\end{array}$$

with exact rows and columns, where the vertical arrows are the natural induced homomorphisms.

We saw in the proof of Proposition 2.6 that $\Omega(\tilde{X})\varepsilon_2 A \overset{t}{\leq} \operatorname{rad} \mathcal{P}(X)$, while Lemma 2.4 implies $\Omega(\overline{X})\varepsilon_2 A \overset{t}{\leq} \operatorname{rad} \mathcal{P}(\overline{X})$. So $\Omega(X)\varepsilon_2 A \overset{t}{\leq} \operatorname{rad} \mathcal{P}(X)$ by Lemma 1.7. □

Corollary 2.8. *If $X \in \mathcal{K}$, then X is a Koszul module.*

Theorem 2.9. *Every standard Koszul standardly stratified algebra is Koszul.*

Proof. We prove the theorem by induction on the number of simple modules. Since C_2 is a standard Koszul standardly stratified algebra by Lemma 2.1, C_2 is also Koszul by the induction hypothesis, thus every simple module is in $\mathcal{K}_2$. So by Corollary 2.8, we only need to prove that all simple modules are $S(1)$-Koszul.

As $S^\circ(1) = \overline{\Delta}^\circ(1)$ is in $\mathcal{C}_{A^\circ}$, for an arbitrary $t \geq 1$,

$$\operatorname{Ext}_A^t(S^\circ(1), \hat{S}^\circ) \subseteq \operatorname{Ext}_A^{t-1}(\hat{S}^\circ, \hat{S}^\circ) \cdot \operatorname{Ext}_A^1(S^\circ(1), \hat{S}^\circ).$$

Applying the K-duality functor, we get that

$$\operatorname{Ext}_A^t(\hat{S}, S(1)) \subseteq \operatorname{Ext}_A^1(\hat{S}, S(1)) \cdot \operatorname{Ext}_A^{t-1}(\hat{S}, \hat{S})$$

for all $t \geq 1$, which finishes the proof. □

Remark 2.10. In view of Lemma 2.1, we also obtained that a standard Koszul standardly stratified algebra is also *recursively Koszul* in the sense of [3].

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