Systems of Linear Equation Exercises (EQ)

Solve the following systems of linear equations;

1.

$$\begin{aligned} x + y + 2z &= 9 \\ 2x + 4y - 3z &= 1 \\ 3x + 6y - 5z &= 0 \end{aligned}$$

Solution

$$\begin{bmatrix} 1 & 1 & 2 & |9 \\ 2 & 4 & -3 & |1 \\ 3 & 6 & -5 & |0 \end{bmatrix} R_2 - 2R_1 \begin{bmatrix} 1 & 1 & 2 & | & 9 \\ 0 & 2 & -7 & | & -17 \\ 3 & 6 & -5 & | & 0 \end{bmatrix} R_3 - 3R_1 \begin{bmatrix} 1 & 1 & 2 & | & 9 \\ 0 & 2 & -7 & | & -17 \\ 0 & 3 & -11 & | & -27 \end{bmatrix}$$

$$\frac{1}{2}R_2 \begin{bmatrix} 1 & 1 & 2 & | & 9 \\ 0 & 1 & \frac{7}{2} & | & -\frac{17}{2} \\ 0 & 3 & -11 & | & -27 \end{bmatrix} R_3 - 3R_2 \begin{bmatrix} 1 & 1 & 2 & | & 9 \\ 0 & 1 & -3.5 & | & -8.5 \\ 0 & 0 & -.5 & | & -1.5 \end{bmatrix} -\frac{1}{2}R_3 \begin{bmatrix} 1 & 1 & 2 & | & 9 \\ 0 & 1 & -3.5 & | & -8.5 \\ 0 & 0 & 1 & | & 3 \end{bmatrix}$$

This is called Row Echelon Form. Now we can solve the system of equation by back substitution method as follows.
From the third row: $3 = 3$.
From the second row: $y - 3.5z = -8.5$. By substituting z value, $y = 2$.
From the first row: $x + y + 2z = 9$; after substituting with the values of y, z, then $x = 1$.
In this case there is a unique solution for the system $(1, 2, 3)$.

2.

$$\begin{aligned} x + y + 2z &= 9 \\ 2x + 4y - 5z &= 1 \\ 3x + 5y - z &= 5 \end{aligned}$$

Solution
After operating some arithmetic calculations on the matrix rows,

$$\begin{bmatrix} 1 & 1 & 2 & | & 9 \\ 0 & 2 & -7 & | & -17 \\ 0 & 0 & 0 & | & -5 \end{bmatrix}$$

From the third row: This is impossible that $0 = -5$. That means there is no solution for this system of linear equations.

3.

$$\begin{aligned} x + y + 2z &= 9 \\ 2x + 4y - 3z &= 1 \\ 3x + 5y - z &= 10 \end{aligned}$$

Solution

After operating some arithmetic calculations on the matrix rows,

$$\begin{bmatrix} 1 & 1 & 2 & | & 9 \\ 0 & 1 & -3.5 & | & -8.5 \end{bmatrix}$$

In this case we ended up with two equations in three variables that can be written as follows,

$$\begin{aligned} x + y + 2z &= 9 \\ y - 3.5z &= 8.5 \end{aligned}$$

Now, let $z = t$, whence $y = 3.5t - 8.5$.
Then we substitute to get the value of x which is $x = 1.5t + 17.5$.
In this case there are infinitely many solutions. The solution set can be written as: $\{(17.5 - 5.5t, 3.5t - 8.5, t) : t \in \mathbb{R}\}$.

4.

$$\begin{aligned} -2x_3 + 7x_5 &= 12 \\ 2x_1 + 4x_2 - 10x_3 + 6x_4 + 12x_5 &= 28 \\ 2x_1 + 4x_2 - 5x_3 + 6x_4 - 5x_5 &= -1 \end{aligned}$$

Solution
After operating some arithmetic calculations on the matrix rows,

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & | & 14 \\ 0 & 0 & 1 & 0 & -3.5 & | & -6 \\ 0 & 0 & 0 & 0 & 1 & | & 2 \end{bmatrix}$$

From the third row: $x_5 = 2$.
From the second row:$x_3 - 3.5_5 = -6 = -6$.
From the first row: $x_1 + 2x_2 - 5x_3 + 3x_4 + 6x_5 = 14, X_1 = 7 - 2x_2 - 3x_4$.
In this case, there are three equations for five variables, so $5 - 3 = 2$ free parameters are to be introduced. These (t and s) are assigned to the variables x_2 and x_4, which are not in the echelon.
Therefore, $x_1 = 7 - 2t - 4s, x_2 = t, x_3 = 1, x_4 = s, x_5 = 2$. In this case

there are infinitely many solutions and the solution set can be written as $\{(7-2t-4s, t, 1, s, 2) : t, s \in \mathbb{R}\}$.

Homogeneous System of Linear Equation Exercises

5.
$$\begin{aligned} 5x_1 + 2x_2 + 6x_3 &= 0 \\ -2x_1 + x_2 + 3x_3 &= 0 \end{aligned}$$

Solution
After operating some arithmetic calculations on the matrix rows,

$$\begin{bmatrix} 1 & -0.5 & -1.5 & | & 0 \\ 0 & 1 & 3 & | & 0 \end{bmatrix}$$

From the second row: $x_2 = -3x_3$. From the first row: $x_1 = o.5x_2 + 1.5x_3$.
There is only one free parameter which is $x_3 = t$. There are infinitely many solutions with the solutions set $\{(0, -3t, t) : t \in \mathbb{R}\}$.

6.
$$\begin{aligned} 2x_1 + x_2 + 3x_3 &= 0 \\ x_1 + 2x_2 &= 0 \\ x_2 + x_3 &= 0 \end{aligned}$$

Solution
After operating some arithmetic calculations on the matrix rows,

$$\begin{bmatrix} 1 & 2 & 0 & | & 0 \\ 0 & 1 & 1 & | & 0 \\ 0 & 0 & 1 & | & 0 \end{bmatrix}$$

In this case, there is a unique solution $x_1 = 0, x_2 = 0, x_3 = 0$. It is called trivial solution, which is always a solution of a homogeneous system.

7. For which values of the parameter a will the following system have one solution, no solution, or infinitely many solutions?

$$\begin{aligned} x + 2y - 3z &= 4 \\ 3x - y + 3z &= 2 \\ 4x + y + (a^2 - 14)z &= a + 2 \end{aligned}$$

Solution

After operating some arithmetic calculations on the matrix rows,

$$\begin{bmatrix} 1 & 2 & -3 & | & 4 \\ 0 & 1 & -2 & | & \frac{10}{7} \\ 0 & 0 & (a^2-16) & | & a-4 \end{bmatrix}$$

If $(a^2 - 16) = (a-4)(a+4) \neq 0$ (i.e., $a \neq \pm 4$, then there is a unique solution.

If $a = 4$, then the last row of the matrix will have all zeros, which means that there are infinitely many solutions.

If $a = -4$, then the last row of the matrix implies $0 = -8$, which is a contradiction, so there is no solution for this system.