Exercise 2 : Matrices

To multiply two matrices (AB), notice that the number of columns of matrix A must be equal to the number of rows of matrix B. Otherwise, the multiplication doesn't exist. For example,

Let $A = \begin{bmatrix} 0 & 1 & 3 \\ 2 & -1 & 1 \\ 3 & 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 5 \\ -1 & 2 \\ 0 & 4 \end{bmatrix}$

Find AB, $B^T A$, AB^T and BA. Solution.

$$AB = \begin{bmatrix} -1 & 14 \\ 3 & 12 \\ 3 & 15 \end{bmatrix}$$

$$B^T A = \begin{bmatrix} -2 & 2 & 2 \\ 16 & 3 & 17 \end{bmatrix},$$

while AB^T and BA do not exist.

Determinants

Find the determinant of the following matrix:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ -4 & 8 & 6 \\ 7 & -8 & 9 \end{bmatrix}.$$

There are three methods to find the determinant.

1. The set of permutations:

$$\det(A) = \begin{vmatrix} 1 & 2 & 3 & | & 1 & 2 \\ -4 & 8 & 6 & | & -4 & 8 \\ 7 & -8 & 9 & | & 7 & -8 \end{vmatrix}$$

$$= (1 \cdot 8 \cdot 9) + (2 \cdot 6 \cdot 7 + (3 \cdot -4 \cdot -8) - (3 \cdot 8 \cdot 7) - (1 \cdot 6 \cdot -8) - (2 \cdot -4 \cdot 9) = 204$$

2. Elementary row transformations (resembles Gaussian elimination):

$$\det(A) = \begin{vmatrix} 1 & 2 & 3 \\ -4 & 8 & 6 \\ 7 & -8 & 9 \end{vmatrix} = R_2{+}4R_1 \begin{vmatrix} 1 & 2 & 3 \\ 0 & 16 & 18 \\ 7 & -8 & 9 \end{vmatrix} R_3{-}7R_1 \begin{vmatrix} 1 & 2 & 3 \\ 0 & 16 & 18 \\ 0 & -22 & -12 \end{vmatrix} R_2/2$$

$$= 2 \begin{vmatrix} 1 & 2 & 3 \\ 0 & 8 & 9 \\ 0 & -22 & -12 \end{vmatrix} R_3/-2 = 2(-2) \begin{vmatrix} 1 & 2 & 3 \\ 0 & 8 & 9 \\ 0 & 11 & 6 \end{vmatrix} C_3/3 = -4(3) \begin{vmatrix} 1 & 2 & 1 \\ 0 & 8 & 3 \\ 0 & 11 & 2 \end{vmatrix}$$

$$= -12 \begin{vmatrix} 1 & 2 & 1 \\ 0 & 8 & 3 \\ 0 & 11 & 2 \end{vmatrix} R_2/8 = -12(8) \begin{vmatrix} 1 & 2 & 1 \\ 0 & 1 & \frac{3}{8} \\ 0 & 11 & 26 \end{vmatrix} R_3{-}11R_2 = -12(8) \begin{vmatrix} 1 & 2 & 1 \\ 0 & 1 & \frac{3}{8} \\ 0 & 0 & \frac{-17}{8} \end{vmatrix}$$

$$= -12 \cdot 8(1 \cdot 1 \cdot \frac{-17}{8}) = -12 \cdot -17 = 204$$

3. The cofactor expansion: by expand along the last row,

$$\det(A) = \begin{vmatrix} 1 & 2 & 3 \\ -4 & 8 & 6 \\ 7 & -8 & 9 \end{vmatrix} = 7 \cdot C_{31} + (-8) \cdot C_{32} + 9 \cdot C_{33}$$

$$= 7(1) \begin{vmatrix} 2 & 3 \\ 8 & 6 \end{vmatrix} + (-8)(-1) \begin{vmatrix} 1 & 3 \\ -4 & 6 \end{vmatrix} + 9(1) \begin{vmatrix} 1 & 2 \\ -4 & 8 \end{vmatrix}$$

$$= 7(12 - 24) + 8(6 + 12) + 9(8 + 8) = 204$$

Notice that the three methods of getting the determinant give the same answer.