Inverse Matrix and Cramer's Rule Calculations Exercises

1. Let $A = \begin{bmatrix} 1 & 0 & 2 \\ -1 & 0 & 3 \\ 0 & 1 & 0 \end{bmatrix}$. Is A invertable?

   To find the inverse of a matrix, first you should find the determinant. Using the cofactor method by expanding along the last row,

$$\det(A) = \sum_{j=1}^{3} a_{3j}C_{3j} = 0 + 1(-1)^{3+2} \cdot \begin{vmatrix} 1 & 2 \\ -1 & 3 \end{vmatrix} + 0 = -5$$

   Since $det(A) \neq 0, A$ is invertable. To find the inverse of the matrix, you should get the adjoint matrix which is the transpose of the matrix of cofactors of all the matrix entries. As the cofactors are the signed minors, first we find the minors:

$$M_{11} = \begin{vmatrix} 0 & 3 \\ 1 & 0 \end{vmatrix} = -3, M_{12} = \begin{vmatrix} -1 & 3 \\ 0 & 0 \end{vmatrix} = 0, M_{13} = \begin{vmatrix} -1 & 0 \\ 0 & 1 \end{vmatrix} = -1$$

$$M_{21} = \begin{vmatrix} 0 & 2 \\ 1 & 0 \end{vmatrix} = -2, M_{22} = \begin{vmatrix} 1 & 2 \\ 0 & 0 \end{vmatrix} = 0, M_{23} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$$

$$M_{31} = \begin{vmatrix} 0 & 2 \\ 0 & 3 \end{vmatrix} = 0, M_{32} = \begin{vmatrix} 1 & 2 \\ -1 & 3 \end{vmatrix} = 5, M_{33} = \begin{vmatrix} 1 & 0 \\ -1 & 0 \end{vmatrix} = 0$$

$$Adj(A) = \begin{bmatrix} +M_{11} & -M_{21} & +M_{31} \\ -M_{12} & +M_{22} & -M_{32} \\ +M_{13} & -M_{23} & +M_{33} \end{bmatrix} = \begin{bmatrix} -3 & 2 & 0 \\ 0 & 0 & -5 \\ -1 & -1 & 0 \end{bmatrix}$$

$$A^{-1} = \frac{1}{det(A)} Adj(A) = \begin{bmatrix} \frac{3}{5} & \frac{-2}{5} & 0 \\ 0 & 0 & 1 \\ \frac{1}{5} & \frac{1}{5} & 0 \end{bmatrix}$$

2. Solve the following systems of linear equations using Cramer's Rule method,

$$x_1 + x_2 + 2x_3 = 9$$
$$2x_1 + 4x_2 - 3x_3 = 1$$
$$3x_1 + 6x_2 - 5x_3 = 0$$

First you should get the determinant,

$$det(A) = \begin{vmatrix} 1 & 1 & 2 \\ 2 & 4 & -3 \\ 3 & 6 & -5 \end{vmatrix}$$

Using the cofactor method by expanding along the first row, $det(A) = -1 \neq 0$. Therefore, Cramer's Rule method is applicable.

$$x_1 = \frac{\begin{vmatrix} 9 & 1 & 2 \\ 1 & 4 & -3 \\ 0 & 6 & -5 \end{vmatrix}}{det(A)} = \frac{-1}{-1} = 1$$

$$x_2 = \frac{\begin{vmatrix} 1 & 9 & 2 \\ 2 & 1 & -3 \\ 3 & 0 & -5 \end{vmatrix}}{det(A)} = \frac{-2}{-1} = 2$$

$$x_3 = \frac{\begin{vmatrix} 1 & 1 & 9 \\ 2 & 4 & 1 \\ 3 & 6 & 0 \end{vmatrix}}{det(A)} = \frac{-3}{-1} = 3$$

In this case there is a unique solution for the linear system of equations which is $(1, 2, 3)$.

3. Find the value of a such that the following system of linear equations has a unique solution.

$$\begin{aligned} x + y + az &= 0 \\ x + ay + z &= 0 \\ ax + y + z &= 0 \end{aligned}$$

This is a homogeneous system of linear equations and it has a unique solution if and only if $det(A) \neq 0$.

$$det(A) = \begin{vmatrix} 1 & 1 & a \\ 1 & a & 1 \\ a & 1 & 1 \end{vmatrix}$$

Using the cofactor method by expanding along the first row,

$$det(A) = 1 \cdot \begin{vmatrix} a & 1 \\ 1 & 1 \end{vmatrix} + 1 \cdot (-1)^{1+2} \begin{vmatrix} 1 & 1 \\ a & 1 \end{vmatrix} + a \cdot \begin{vmatrix} 1 & a \\ a & 1 \end{vmatrix}$$

$$= (a-1) - (1-a) + a(1-a^2) = -1(1-a) - (1-a) + a(1-a)(1+a)$$
$$= (1-a)[-1-1+a(1+a)] = (1-a)(a+2)(a-1) = 0$$
$$a = 1, -2$$

Therefore there is a unique solution of this linear system if and only if $a \neq -2$ and $a \neq 1$. Otherwise, there will be infinitely many solutions.