Inner Product Spaces Exercise

Solve the following question.

1. Check whether the vectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ are linearly independent. If yes, then apply the Gram–Schmidt orthogonalization process to them so that to obtain a complete orthonormal basis $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ in $\mathbb{R}^3$: $\mathbf{u}_1 = (1,1,1)$, $\mathbf{u}_2 = (0,1,1)$, $\mathbf{u}_3 = (0,0,1)$.

   Solution.

   $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ are linearly independent since the determinanat of three vector together is $= 1 \neq 0$. So we can apply (G-S) process as follows.

   Step 1 - find $\mathbf{v}_1$:

$$\mathbf{v}_1 = \frac{\mathbf{u}_1}{\|\mathbf{u}_1\|} = \frac{\mathbf{u}_1}{\sqrt{3}} = (\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$$

   ,

   Step 2 - Step 1 - find $\mathbf{v}_1$. First find projection of $\mathbf{u}_2$ onto $\mathbf{v}_1$:

$$\text{proj}_{\mathbf{v}_1}\mathbf{u}_2 = \langle \mathbf{u}_2, \mathbf{v}_1 \rangle \mathbf{v}_1$$

   . But

$$< \mathbf{u}_2, \mathbf{v}_1 >= 0 \cdot \frac{1}{\sqrt{3}} + 1 \cdot \frac{1}{\sqrt{3}} + 1 \cdot \frac{1}{\sqrt{3}} = \frac{2}{\sqrt{3}}$$

   so

$$\text{proj}_{\mathbf{v}_1}\mathbf{u}_2 = \frac{2}{\sqrt{3}}\mathbf{v}_1 = (\frac{2}{3}, \frac{2}{3}, \frac{2}{3})$$

   and

$$\mathbf{v}_2 = \frac{\mathbf{u}_2 - \text{proj}_{\mathbf{v}_1}\mathbf{u}_2}{\|\mathbf{u}_2 - \text{proj}_{\mathbf{v}_1}\mathbf{u}_2\|} = \frac{(\frac{-2}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})}{\sqrt{(\frac{4}{9} + \frac{1}{9} + \frac{1}{9})}} = (\frac{-2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}})$$

   Step 3 - find $\mathbf{v}_3$:

$$\mathbf{v}_3 = \frac{\mathbf{u}_3 - \text{proj}_{lin\{\mathbf{v}_1,\mathbf{v}_2\}}\mathbf{u}_3}{\|\mathbf{u}_3 - \text{proj}_{lin\{\mathbf{v}_1,\mathbf{v}2\}}\mathbf{u}_3\|}$$

   where

$$\text{proj}_{lin\{\mathbf{v}_1,\mathbf{v}_2\}}\mathbf{u}_3 =< \mathbf{u}_3, \mathbf{v}_1 > \mathbf{v}_1 + < \mathbf{u}_3, \mathbf{v}_2 > \mathbf{v}_2 = \frac{1}{\sqrt{3}}\mathbf{v}_1 + \frac{1}{\sqrt{6}}\mathbf{v}_2 = (0, \frac{-1}{2}, \frac{1}{2})$$

so

$$\mathbf{v}_3 = \frac{(0, \frac{-1}{2}, \frac{1}{2})}{\sqrt{(0 + \frac{1}{4} + \frac{1}{4})}} = (0, \frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$$

To check your answer, check $\|\mathbf{v}_1\| = \|\mathbf{v}_2\| = \|\mathbf{v}_3\| = 1$ and that they are pairwise orthogonal.