Eigenvalues and Eigenvectors Exercise

Find the eigenvalues and eigenvectors of the following matrices:

1.

$$\mathbf{A} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

Solution.

To find the eigenvalues, solve the equation $|\mathbf{A} - \lambda I| = 0$

$$\begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)^2 - 1 = (\lambda - 3)(\lambda - 1) = 0$$

Therefore, the eigenvalues are, $\lambda_{\mathbf{1}} = 3, \lambda_{\mathbf{2}} = 1$.

Then to find the eigenvector, for each eigenvalue you should solve the following homogeneous system of linear equation $(\mathbf{A} - \lambda I)\,\mathbf{u}$.
For example, $\lambda_{\mathbf{1}} = 3, (\mathbf{A} - 3I)\,\mathbf{u}$

$$\begin{bmatrix} 2-3 & 1 \\ 1 & 2-3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

By solving this system, you get $x = y$. Therefore the first eigenvector can be written as $\mathbf{u_1} = \begin{bmatrix} x \\ x \end{bmatrix}$, then we normalize it by $\frac{\mathbf{u_1}}{\|\mathbf{u_1}\|}$. Therefore

$$\mathbf{u_1} = \frac{\mathbf{u_1}}{\sqrt{2\dot{x}^2}} = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$$

By repeating the same for $\lambda_{\mathbf{2}} = 1$, the second normalized eigenvector can be written as $\mathbf{u_2} = (\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$.

Notice that $\mathbf{u_1}, \mathbf{u_2}$ together form an orthonormal basis.

2.

$$\mathbf{A} = \begin{pmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{pmatrix}$$

Solution.
Following the same approach, to find the eigenvalue solve,

$$|\mathbf{A} - \lambda I| = 0$$

In this case there are three eigenvalues, $\lambda_{\mathbf{1}} = 8, \lambda_{\mathbf{2}} = 2, \lambda_{\mathbf{3}} = 2$.

To find the eigenvector, for each λ, solve $(\mathbf{A} - \lambda I)\,\mathbf{u}$.

For $\lambda_{\mathbf{1}} = 8, \mathbf{u_1} = (\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$.

For $\lambda_{\mathbf{2}} = 2, \mathbf{u_2} = (\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0)$.

For $\lambda_{\mathbf{3}} = 2, \mathbf{u_3} = (\frac{-1}{\sqrt{6}}, \frac{-1}{\sqrt{6}}, \frac{2}{\sqrt{6}})$.

Notice that for the same eigenvalue, the corresponding eigenvector can be rotated differently. In this case we have what is so called eigen subspace.