Math. A2, LINEAR ALGEBRA
SYSTEMS OF LINEAR EQUATIONS
Lessons 1-3.
Gaussian and Gauss-Jordan Elimination and Matrices

- **Definition**: Augmented matrix of system of linear equations is a matrix consisting of the coefficients of the unknowns and the constants, i.e. the augmented matrix of the following system of equations:
$$\begin{array}{ccccccc} a_{11}x_1 & + & a_{12}x_2 & + & a_{13}x_3 & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & a_{23}x_3 & = & b_2 \\ a_{31}x_1 & + & a_{32}x_2 & + & a_{33}x_3 & = & b_3 \end{array} \text{ is } \left(\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{array}\right)$$

- **Definition**: Gaussian Elimination is a procedure for solving systems of linear equations. It is based on the idea of reducing the augmented matrix to a simple form so that the system of equations can be solved by inspection.

- **The procedure**:

  - **Step 1.** Locate the left most column that does not consist entirely of zeros.
  - **Step 2.** Interchange the top row with another row, if necessary, to bring a nonzero entry to the top of the column found in Step1.
  - **Step 3.** If the entry that is now at the top of the column found in Step 1 is a, multiply the first row by $1/a$ in order to introduce a leading 1.
  - **Step 4.** Add suitable multiples of the top row to the rows below so that all entries below leading 1 become zeros.
  - **Step 5.** Cover the top row in the matrix and begin again with Step 1 applied to the sub matrix that remains. Continue in this way until the entire matrix is in *row-echelon form* (all entries in a column below a leading 1 are zeros).

- **Types of solution.** In the matrix of Step 5, which is in the row-echelon form:

  - If the last row is all zeros, then there are infinite number of solutions.
  - If the last row is all zeros except the constant value (on the right hand side), then it is contradiction and there is no solution.
  - Otherwise, there is exactly one solution.

Transforming this resulted matrix into a system of linear equations makes it easy to get the solution(s) by back-substitution.

No back-substitution is needed if you do the following.

- **Definition**: Gauss-Jordan elimination has the same procedure as Gaussian Elimination with additional steps, in which zeros are being introduced above the leading 1's.