Math. A2, Lessons 4-5.
Determinants, inverses, and Cramér's rule

- **Definition**: Determinant of a square matrix $\mathbf{A}$ ($\det(\mathbf{A}) = |\mathbf{A}|$) is a number assigned to the matrix $\mathbf{A}$ in the following way:

  $\det(\mathbf{A}) = |\mathbf{a}_1 \cdots \mathbf{a}_n| = \pm$Volume of the parallelepiped stretched by $\mathbf{a}_1, \dots, \mathbf{a}_n$

- The Determinant of $\mathbf{A}$ can be found by one of the following methods:
  - $\det(\mathbf{A}) = \sum_\sigma (-1)^{I(\sigma)} a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)}$, where σ is the set of permutations of $\{1, 2, \dots, n\}$, and $I(\sigma)$ is the number of inversions in the permutation σ.
  - As the product of the entries in the main diagonal of the matrix $\mathbf{A}$ in row-echelon form (which is an upper triangular matrix) obtained by Gaussian elimination.
  - By Co-factor expansion: if M_{ij} is the determinant of the $(n-1) \times (n-1)$ matrix obtained from $\mathbf{A}$ after removing its row i and column j, and $C_{ij} = (-1)^{i+j} M_{ij}$, then we have this **theorem:**
    - $\det(\mathbf{A}) = \sum_{j=1}^n a_{ij} C_{ij}$, $i = 1, \dots, n$ (expand along row i)
    - $\det(\mathbf{A}) = \sum_{i=1}^n a_{ij} C_{ij}$, $j = 1, \dots, n$ (expand along column j)
    - $\sum_{j=1}^n a_{ij} C_{kj} = 0$ if $i \neq k$ (Skew expansion)

- The determinant of an upper triangular, lower triangular and diagonal matrix is the product of its diagonal entries.

- $\det(\mathbf{A}^T) = \det(\mathbf{A})$.

- If two columns (rows) in $\mathbf{A}$ are interchanged, then its determinant will be multiplied by (-1).

- If a column (row) of an $n \times n$ matrix $\mathbf{A}$ is multiplied by $c \in \mathbb{R}$, then its determinant will be multiplied by c, consequently, $\det(c\mathbf{A}) = c^n \det(\mathbf{A})$.

- If a constant multiple of a row (column) of $\mathbf{A}$ is added to another one, then $\det(\mathbf{A})$ will not be changed.

- $\det(\mathbf{AB}) = \det(\mathbf{A}) \cdot \det(\mathbf{B}) = \det(\mathbf{BA})$.

- If $\mathbf{A}$ is invertible, then $\det(\mathbf{A}^{-1}) = \frac{1}{\det(\mathbf{A})} \neq 0$.

- **Definition**: The adjoint matrix of $\mathbf{A}$ is

$$\operatorname{adj}(\mathbf{A}) = \begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ \vdots & \vdots & & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{bmatrix}^T = \begin{bmatrix} C_{11} & \cdots & C_{n1} \\ C_{12} & \cdots & C_{n2} \\ \vdots & & \vdots \\ C_{1n} & \cdots & C_{nn} \end{bmatrix}$$

- **Theorem**: If $\mathbf{A}$ is invertible, then $\mathbf{A}^{-1} = \frac{1}{\det(\mathbf{A})}\operatorname{adj}(\mathbf{A})$.

  Proof: Use the direct and the skew cofactor expansion formulas

$$\sum_{j=1}^{n} a_{ij} C_{ij} = \det(\mathbf{A}), \quad \forall i = 1, \dots, n$$

  and

$$\sum_{j=1}^{n} a_{ij} C_{kj} = 0, \quad \forall i \neq k.$$

- **Vandermonde determinant**:

$$\begin{vmatrix} 1 & x_1 & \cdots & x_1^{n-1} \\ \vdots & \vdots & & \vdots \\ 1 & x_n & \cdots & x_n^{n-1} \end{vmatrix} = \prod_{1 \le i < j \le n} (x_j - x_i)$$

- **Properties of the matrix inversion:** If $\mathbf{A}$ and $\mathbf{B}$ are $n \times n$ invertible matrices, then
  - $\mathbf{AB}$ is also invertible and $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$
  - $(\mathbf{A}^{-1})^T = (\mathbf{A}^T)^{-1}$
  - $(c\mathbf{A})^{-1} = \frac{1}{c}\mathbf{A}^{-1}, \quad c \in \mathbb{R}$
  - $(\mathbf{A}^{-1})^{-1} = \mathbf{A}$.

- **Cramér's rule**: If $\mathbf{Ax} = \mathbf{b}$, and $\mathbf{A}$ is invertible, then $x_i = \frac{\det(\mathbf{A}_i)}{\det(\mathbf{A})}$, $i = 1, \cdots, n$, where $\mathbf{A}_i$ is the matrix obtained from $\mathbf{A}$ by replacing its ith column by the vector $\mathbf{b}$.