**Math. A2**
**Vector Spaces**

- **Definition**: The non-empty set $V$ is a vector space over $\mathbb{R}$ if for $\mathbf{v}, \mathbf{u} \in V$ the following operations are defined:
  - $\mathbf{v} + \mathbf{u} \in V$.
  - $k\mathbf{v} \in V$, for any $k \in \mathbb{R}$.

  Further these operations have the following properties:
  - $\mathbf{v} + \mathbf{u} = \mathbf{u} + \mathbf{v}$.
  - $(\mathbf{v} + \mathbf{u}) + \mathbf{w} = \mathbf{v} + (\mathbf{u} + \mathbf{w})$.
  - $\mathbf{0} \in V$(zero element) such that $\mathbf{v} + \mathbf{0} = \mathbf{v}$.
  - for any $\mathbf{v}$ there exists $-\mathbf{v}$ such that $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$.
  - $(kc)\mathbf{v} = k(c\mathbf{v}) = c(k\mathbf{v})$, for any $k, c \in \mathbb{R}$.
  - $1 \cdot \mathbf{v} = \mathbf{v}$.
  - $0 \cdot \mathbf{v} = \mathbf{0}$.
  - $k(\mathbf{u} + \mathbf{v}) = k\mathbf{u} + k\mathbf{v}$ and $(\mathbf{u} + \mathbf{v})k = \mathbf{u}k + \mathbf{v}k$.
  - $(k + c)\mathbf{u} = k\mathbf{u} + c\mathbf{u}$ and $\mathbf{u}(k + c) = \mathbf{u}k + \mathbf{u}c$.
- **Definition**: Let $V$ be a vector space. The subset $W \in V$ is a subspace of V if it is closed under operations in V: if $\mathbf{u}, \mathbf{v} \in W$, then $\mathbf{u} + \mathbf{v} \in W$ and if $\mathbf{u} \in W$, $k \in \mathbb{R}$ then $k\mathbf{u} \in W$.
- **Remark**:
  - $\mathbf{0} \in W$, because $\mathbf{u} \in W$, $-\mathbf{u} \in W \Rightarrow \mathbf{u} + (-\mathbf{u}) \in W \Rightarrow \mathbf{0} \in W$
  - $V, \{\mathbf{0}\}$ are trivial subspaces.
- **Definition**: Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \in V$, where V is a vector space. The subspace (linear space) spanned by $\mathbf{v}_1, \ldots, \mathbf{v}_n$ is the following: $\mathrm{lin}\{\mathbf{v}_1, \ldots, \mathbf{v}_n\} = \{\mathbf{v} = k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \cdots + k_n\mathbf{v}_n, \text{with some } k_1, k_2, \ldots, k_n \in \mathbb{R}\} \subset V$.
- **Definition**: $\mathbf{v}_1, \ldots, \mathbf{v}_n$ generate $V$ if $\mathrm{lin}\{\mathbf{v}_1, \ldots, \mathbf{v}_n\} = V$
- **Definition**: The vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n \in V$ are linearly independent if $k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \cdots + k_n\mathbf{v}_n = \mathbf{0} \Rightarrow k_1 = k_2 = \cdots k_n = 0$. Otherwise, they are linearly dependent.
- **Definition**: $\mathbf{b}_1, \ldots, \mathbf{b}_n \in V$ form a basis of V if:

  - $\mathbf{b}_1, \ldots, \mathbf{b}_n$ are linearly independent,
  - $\text{lin}\{\mathbf{b}_1, \ldots, \mathbf{b}_n\} = V$.

- **Remarks**:

  - If $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are linearly independent, then none of them can be written as a linear combination of the others.
  - Basis is a linear independent set which generates $V$.
  - There may be several bases in $V$, but the number of vectors forming a basis in $V$ is unique. It is called the dimension of $V$.