Math. A2
Linear dependance / independence

- The vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m \in \mathbb{R}^n$ are *linearly independent* if $k_1\mathbf{a}_1 + k_2\mathbf{a}_2 + \ldots + k_m\mathbf{a}_m = \mathbf{0}$ implies that $k_1 = k_2 = \ldots = k_m = 0$ (only their trivial linear combination is 0). Otherwise, they are linearly dependent.

- Let $\mathbf{a} \in \mathbb{R}$, then the standard basis is

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \ldots, \mathbf{e}_n = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$$

and $\mathbf{a}$ can be written as $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + \cdots + a_n\mathbf{e}_n$

- For $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m \in \mathbb{R}^n$
  - If $m > n$, then they are dependent.
  - If $m = n$, then they are independent $\Leftrightarrow |\mathbf{a}_1\mathbf{a}_2 \cdots \mathbf{a}_n| \neq 0$.
  - If $m < n$, then decide by Gaussian elimination.

- **Definition**: Let $\mathbf{A}$ be an $n \times m$ matrix. The rank of $\mathbf{A}$ is the maximum number of its linearly independent rows, which is the same as the maximum number of its linearly independent columns, i.e. $\text{rank}(\mathbf{A}) \leq \min\{n, m\}$.

- $\text{lin}\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m, \mathbf{0}\} = \text{lin}\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m\}$.

- Consider the system of linear equations $\mathbf{Ax} = \mathbf{b}$, where $\mathbf{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$ and $\mathbf{A} = (\mathbf{a}_1, \ldots, \mathbf{a}_m)$. Then

$$\mathbf{Ax} = \begin{pmatrix} \mathbf{a}_1 & \cdots & \mathbf{a}_m \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix} = x_1\mathbf{a}_1 + \cdots + x_m\mathbf{a}_m \in \text{lin}\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m\}.$$

- If $\mathbf{b} \notin \operatorname{lin}\{\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_m\}$, then there is no solution. Since

  $$\operatorname{rank}(\mathbf{A}) \neq \operatorname{rank}(\mathbf{A}|\mathbf{b}),$$

  the system is *inconsistent*.
- If $\mathbf{b} \in \operatorname{lin}\{\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_m\}$, then there is solution, i.e., the system is *consistent*.
  * If $\operatorname{rank}(\mathbf{A}) = \operatorname{rank}(\mathbf{A}|\mathbf{b}) = m$, then there is a unique solution.
  * If $r = \operatorname{rank}(\mathbf{A}) = \operatorname{rank}(\mathbf{A}|\mathbf{b}) < m$, then there are infinitely many solutions expressed by $m - r$ free parameters.
  * Special case: If $n = m$, then $\operatorname{rank}(\mathbf{A}) = n \Leftrightarrow \det(\mathbf{A}) \neq 0$ and in this case $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$ and can be found by the Cramer's rule.
  * Special case: If $n = m$ and $\mathbf{b} = \mathbf{0}$ (homogeneous system), then we always have a solution ($\mathbf{x} = \mathbf{0}$ is always a solution, called trivial). There are non-trivial solutions too (they are infinitely many) if and only if $|\mathbf{A}| = 0$.