Math. A2
Inner Product Spaces

- **Definition**: The real vector space V is a real inner product space if a new operation (inner product) is defined between its vectors in the following way:
$$V \times V \to \mathbb{R}$$
$$< \mathbf{v}, \mathbf{u} > \in \mathbb{R} \text{ for any } \mathbf{u}, \mathbf{v} \in V$$
with the following properties:
  - $< \mathbf{v}, \mathbf{u} > = < \mathbf{u}, \mathbf{v} >, \forall \mathbf{u}, \mathbf{v} \in V$.
  - $< \mathbf{v} + \mathbf{w}, \mathbf{u} > = < \mathbf{v}, \mathbf{u} > + < \mathbf{w}, \mathbf{u} >$.
  - $< k\mathbf{v}, \mathbf{u} > = k < \mathbf{v}, \mathbf{u} >, k \in \mathbb{R}$.
  - $< \mathbf{v}, \mathbf{v} > \geq 0$ and $< \mathbf{v}, \mathbf{v} > = 0 \Leftrightarrow \mathbf{v} = \mathbf{0}$
- **Definition**: $\mathbf{v}_1, \ldots, \mathbf{v}_m \in \mathbb{R}^n$ form an orthogonal system if $< \mathbf{v}_i, \mathbf{v}_j > = 0$, where $i \neq j, m \leq n$.
- **Proposition**: An orthogonal system consists of linearly independent vectors, $(m \leq n)$.
- **Definition**: $\mathbf{v}_1, \ldots, \mathbf{v}_m \in \mathbb{R}^n$ $(m \leq n)$ form an orthonormal system if
$$< \mathbf{v}_i, \mathbf{v}_j > = \delta_{ij} = \begin{cases} 1 & \text{if} \quad i = j \quad \text{orthogonal system of unit length vectors} \\ 0 & \text{if} \quad i \neq j \quad < \mathbf{v}_i, \mathbf{v}_j > = \|\mathbf{v}_i\|^2 = \mathbf{1} \Rightarrow \|\mathbf{v}_i\| = \mathbf{1} \end{cases}$$
- **Definition**: If $m = n$, then an orthonormal system is complete $\Rightarrow$ $\mathbf{v}_1, \ldots, \mathbf{v}_n \in \mathbb{R}^n$ is an orthonormal basis.
- **Fact**: a finite dimensional inner product space always has az orthonormal basis.
- **Definition**: Let $V = \mathbb{R}^n$, and $U \subset V$ be a proper subspace of V, $U = \text{lin}\{\mathbf{u}_1, \ldots, \mathbf{u}_m\}$ $(m < n)$, $\mathbf{u}_1, \ldots, \mathbf{u}_m$ are linearly independent and assume that they are from an orthonormal system, then
$$\mathbf{v}_1 = \text{proj}_U \mathbf{v} = < \mathbf{v}, \mathbf{u}_1 > \mathbf{u}_1 + < \mathbf{v}, \mathbf{u}_2 > \mathbf{u}_2 + \cdots + < \mathbf{v}, \mathbf{u}_m > \mathbf{u}_m$$
is the projection of the vector $\mathbf{v}$ on the subspace U. $\mathbf{v} - \mathbf{v}_1$ is the orthogonal component of $\mathbf{v}$ on U. Let $\mathbf{u}_1, \ldots, \mathbf{u}_m \in \mathbb{R}^n$ be linearly independent non-zero vectors $(m \leq n)$, then there exist an orthonormal system $\mathbf{v}_1, \ldots, \mathbf{v}_m$ such that $\text{lin}\{\mathbf{u}_1, \ldots, \mathbf{u}_m\} = \text{lin}\{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$. This

is obtained by the following process (**Gram–Schmidt orthogonalization**):

$$\mathbf{v}_1 = \frac{\mathbf{u}_1}{\|\mathbf{u}_1\|},$$

if we already have $\mathbf{v}_1, \ldots, \mathbf{v}_{k-1}$, then

$$\mathbf{v}_k = \frac{\mathbf{u}_k - \sum_{i=1}^{k-1} \langle \mathbf{u}_k, \mathbf{v}_i \rangle \mathbf{v}_i}{\|\mathbf{u}_k - \sum_{i=1}^{k-1} \langle \mathbf{u}_k, \mathbf{v}_i \rangle \mathbf{v}_i\|}, \quad k = 2, \ldots, n.$$

- **Cauchy–Schwarz inequality**:

  $$|\langle \mathbf{a}, \mathbf{b} \rangle| \leq \|\mathbf{a}\| \cdot \|\mathbf{b}\|$$

  or equivalently,

  $$\langle \mathbf{a}, \mathbf{b} \rangle^2 \leq \|\mathbf{a}\|^2 \cdot \|\mathbf{b}\|^2.$$

- The *length* of $\mathbf{a}$: $\|\mathbf{a}\| = \langle \mathbf{a}, \mathbf{a} \rangle^{\frac{1}{2}}$

- The *distance* between $\mathbf{a}$ and $\mathbf{b}$: $d(\mathbf{a}, \mathbf{b}) = \|\mathbf{a} - \mathbf{b}\|$

- For the *angle* γ between $\mathbf{a} \neq \mathbf{0}$ and $\mathbf{b} \neq \mathbf{0}$: $\cos\gamma = \frac{\langle \mathbf{a}, \mathbf{b} \rangle}{\|\mathbf{a}\| \cdot \|\mathbf{b}\|}$, by the cosine rule. In view of the Cauchy–Schwarz inequality, $|\cos\gamma| \leq 1$ and $\cos\gamma = 0$ (where γ is enclosed by non-zero vectors) if and only if $\langle \mathbf{a}, \mathbf{b} \rangle = 0$, and ve use the notation $\mathbf{a} \perp \mathbf{b}$ for this.

- **Generalized Pythagorean theorem**: If $\mathbf{a} \perp \mathbf{b}$, i.e., if $\langle \mathbf{a}, \mathbf{b} \rangle = 0$, then $\|\mathbf{a} + \mathbf{b}\|^2 = \|\mathbf{a}\|^2 + \|\mathbf{b}\|^2$

- **Best Approximation Theorem**: If U is a finite dimensional subspace of an inner product space V, then for any $\mathbf{v} \in V$:

  $$\|\mathbf{v} - \mathrm{proj}_U \mathbf{v}\| \leq \|\mathbf{v} - \mathbf{w}\|$$

  $\forall \mathbf{w} \in U$.