Determinants, Cramer's rule, Inverse matrix

Mathematics A2

5th week

1. Let $\mathbf{A}=\begin{bmatrix}\cos x & -\sin x\\ \sin x & \cos x\end{bmatrix}$, find

a.) $\mathbf{A}^2$ b.) $\mathbf{A}^{2012}$ c.) $\det\mathbf{A}$ d.) $\mathbf{A}^{-1}$ e.) $\mathbf{A}^{-2012}$

2. Let $\mathbf{A}=\begin{bmatrix}1 & 0 & 2\\ -1 & 0 & 3\\ 0 & 1 & 0\end{bmatrix}$, find

a.) $\det\mathbf{A}$ b.) $\det(3\mathbf{A})$ c.) $\det(\mathbf{A}^{-1})$ d.) $\det(3\mathbf{A}^{-1})$ e.) $\det(\mathbf{A}^{T})$

3. Assume that $\det\mathbf{A}=-2$, where $\mathbf{A}$ is a square matrix of 3x3. Give the value of

a.) $\det(\mathbf{A}^2)=$ b.) $\det(\mathbf{A}^{-1})=$ c.) $\det((5\mathbf{A})^{-1})=$

4. $\begin{vmatrix}\sin x & \cos x & -\sin x\\ -\cos x & \sin x & \cos x\\ 1 & 1 & 1\end{vmatrix}=$

5. $\begin{vmatrix}2 & 1 & 3 & 1\\ 1 & 0 & 1 & 1\\ 0 & 2 & 1 & 0\\ 0 & 1 & 2 & 3\end{vmatrix}=$

4. Find the value of k (if possible) such that the matrix be invertible?

a.) $\mathbf{A}=\begin{bmatrix}k-1 & 2\\ 2 & k-1\end{bmatrix}$ b.) $\mathbf{B}=\begin{bmatrix}1 & 2 & 4\\ 3 & 1 & 6\\ k & 3 & 2\end{bmatrix}$

5. For which value(s) of k will the matrix $\mathbf{A}=\begin{bmatrix}k+1 & 2 & 4\\ k & 1 & 6\\ k-1 & 3 & 3\end{bmatrix}$ fail to be invertible?

6. Use row reduction to show that $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = (b-a)(c-a)(c-b)$.

7. Using the adjoint matrix find the inverse of the matrix:

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 2 \\ 4 & 4 & 3 \\ 3 & 7 & 6 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 3 & 0 & 0 \\ 9 & 1 & 0 \\ -4 & 2 & 4 \end{bmatrix} \qquad \mathbf{C} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

8. Let **A** and **B** be *nxn* matrices. Show that if **A** is invertible, then $\det \mathbf{B} = \det(\mathbf{A}^{-1}\mathbf{BA})$.

9. Solve the following systems *using Cramer's rule:*

a.) $\begin{aligned} 2x - y + z &= 8 \\ 4x + 3y + z &= 7 \\ 6x + 2y + 2z &= 15 \end{aligned}$

b.) $\begin{aligned} x_1 + x_2 + 2x_3 &= 8 \\ -x_1 - 2x_2 + 3x_3 &= 1 \\ 3x_1 - 7x_2 + 4x_3 &= 10 \end{aligned}$

c.) $\begin{aligned} 2x_1 - x_2 + x_3 - 4x_4 &= -32 \\ 7x_1 + 2x_2 + 9x_3 - x_4 &= 14 \\ 3x_1 - x_2 + x_3 + x_4 &= 11 \\ x_1 + x_2 - 4x_3 - 2x_4 &= -4 \end{aligned}$

10. Solve the system using the formula $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$:

$$\begin{aligned} x_1 + 2x_2 &= 7 \\ 2x_1 + 5x_2 &= -3 \end{aligned}$$

11. Solve the matrix equation $\begin{bmatrix} -1 & 5 & 1 \\ 0 & 1 & 2 \\ -1 & 0 & 1 \end{bmatrix} \cdot \mathbf{X} = \begin{bmatrix} 0 & -4 \\ 2 & 5 \\ 0 & 1 \end{bmatrix}$.

12. Answer if the following statements are true or false. Give reason for your answer. (If true, say why, if false, give such an example.)

a.) If **AB** and **BA** are both defined then they (both products) have the same size.

b.) If $\mathbf{AB} = \mathbf{0}$ then at least one of the factors is a zero matrix.

c.) If the homogeneous system $\mathbf{Ax} = \mathbf{0}$ contains more equations than unknowns then its only solution is the trivial solution $\mathbf{x} = \mathbf{0}$.

d.) The product of two invertible matrices (of the same size) is also invertible.