Formulas (to differential equations)

Math. A3, Midterm Test I.

$\sin^2 x + \cos^2 x = 1$

$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$

$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$

$\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x \cdot \tan y}$

$\sin 2x = 2 \sin x \cos x$

$\cos 2x = \cos^2 x - \sin^2 x$

$\tan 2x = \frac{2 \tan x}{1 - \tan^2 x}$

$\sin^2 x = \frac{1 - \cos 2x}{2}$, $\quad \cos^2 x = \frac{1 + \cos 2x}{2}$

$\sin x + \sin y = 2 \sin \frac{x+y}{2} \cos \frac{x-y}{2}$

$\sin x - \sin y = 2 \cos \frac{x+y}{2} \sin \frac{x-y}{2}$

$\cos x + \cos y = 2 \cos \frac{x+y}{2} \cos \frac{x-y}{2}$

$\cos x - \cos y = -2 \sin \frac{x+y}{2} \sin \frac{x-y}{2}$

$\sin x \cos y = \frac{1}{2}[\sin(x+y) + \sin(x-y)]$

$\cos x \cos y = \frac{1}{2}[\cos(x+y) + \cos(x-y)]$

$\sin x \sin y = -\frac{1}{2}[\cos(x+y) - \cos(x-y)]$

$\sinh x = \frac{e^x - e^{-x}}{2}$, $\quad \cosh x = \frac{e^x + e^{-x}}{2}$

$\cosh^2 x - \sinh^2 x = 1$

$\sinh 2x = 2 \sinh x \cosh x$

$\cosh 2x = \cosh^2 x + \sinh^2 x$

$\cosh^2 x = \frac{\cosh 2x + 1}{2}$, $\quad \sinh^2 x = \frac{\cosh 2x - 1}{2}$

derivatives:

$(\sinh x)' = \cosh x$

$(\cosh x)' = \sinh x$

$(\log_a x)' = \frac{1}{x \ln a}$

$(x^\alpha)' = \alpha x^{\alpha-1}$

$(e^x)' = e^x$

$(a^x)' = a^x \ln(a)$

$(\sin x)' = \cos x$

$(\cos x)' = -\sin x$

$(\tan x)' = \frac{1}{\cos^2 x}$

$(\cot x)' = -\frac{1}{\sin^2 x}$

$(\ln x)' = \frac{1}{x}$

$(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$

$(\arctan x)' = \frac{1}{1+x^2}$

$(\operatorname{ar\,sinh} x)' = \frac{1}{\sqrt{1+x^2}}$

$(\operatorname{ar\,cosh} x)' = \frac{1}{\sqrt{x^2-1}}$

$(\operatorname{ar\,tanh} x)' = \frac{1}{1-x^2}$

$(\operatorname{ar\,coth} x)' = \frac{1}{1-x^2}$

$(\arccos x)' = -\frac{1}{\sqrt{1-x^2}}$

$(\operatorname{arc\,cot} x)' = -\frac{1}{1+x^2}$

differentiation rules:

$(cu)' = cu'$ $\quad$ (c is constant)

$(u+v)' = u' + v'$

$(uv)' = u'v + uv'$

$\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$

$\frac{d}{dx} f(g(x)) = \frac{df}{dg} \frac{dg}{dx}$

integration rules:

$\int cf \, dx = c \int f \, dx$ $\quad$ (c is constant)

$\int (f+g) \, dx = \int f \, dx + \int g \, dx$

$\int f(ax+b) \, dx = \frac{1}{a} F(ax+b) + c$,

where F is antiderivative of f

$\int f(g(x)) g'(x) \, dx = F(g(x)) + c$,

where F is antiderivative of f

$\int f^\alpha f' \, dx = \frac{f^{\alpha+1}}{\alpha+1} + c$, if $\alpha \neq -1$

$\int \frac{f'}{f} \, dx = \ln|f| + c$

$\int uv' \, dx = uv - \int u'v \, dx$

notable substitutions:

$R(e^x)$	$e^x = t$
$R(\sqrt{ax+b})$	$\sqrt{ax+b} = t$
$R\left(\frac{\sqrt{ax+b}}{\sqrt{cx+d}}\right)$	$\frac{\sqrt{ax+b}}{\sqrt{cx+d}} = t$
$R(\sin x, \cos x)$	$\sin x, \cos x, \tan x, \tan \frac{x}{2} = t$
$R(x, \sqrt{a^2 - x^2})$	$x = a \sin t, \; x = a \cos t$
$R(x, \sqrt{a^2 + x^2})$	$x = a \sinh t$
$R(x, \sqrt{x^2 - a^2})$	$x = a \cosh t$

antiderivatives:

$\int x^\alpha \, dx = \frac{x^{\alpha+1}}{\alpha+1} + c$ $\quad$ $(\alpha \neq -1)$

$\int e^{ax} \, dx = \frac{1}{a} e^{ax} + c$

$\int a^x \, dx = \frac{a^x}{\ln a} + c$

$\int \cos x \, dx = \sin x + c$

$\int \sin x \, dx = -\cos x + c$

$\int \frac{1}{\cos^2 x} \, dx = \tan x + c$

$\int \frac{1}{\sin^2 x} \, dx = -\cot x + c$

$\int \frac{1}{x} \, dx = \ln|x| + c$

$\int \frac{dx}{\sqrt{a^2 - x^2}} = \arcsin \frac{x}{a} + c$

$\int \frac{dx}{x^2 + a^2} = \frac{1}{a} \arctan \frac{x}{a} + c$

$\int \frac{dx}{\sqrt{x^2 + a^2}} = \operatorname{ar\,sinh} \frac{x}{a} + c$

$\int \frac{dx}{\sqrt{x^2 - a^2}} = \operatorname{ar\,cosh} \frac{x}{a} + c$

$\int \frac{dx}{a^2 - x^2} = \frac{1}{a} \operatorname{ar\,tanh} \frac{x}{a} + c$, $\quad$ if $\left|\frac{x}{a}\right| < 1$

$\int \frac{dx}{a^2 - x^2} = \frac{1}{a} \operatorname{ar\,coth} \frac{x}{a} + c$, $\quad$ if $\left|\frac{x}{a}\right| > 1$

$\int \tan x \, dx = -\ln|\cos x| + c$

$\int \cot x \, dx = \ln|\sin x| + c$

1.
$$\mathrm{e}^{it} = \cos t + i \cdot \sin t, \qquad t \in \mathbb{R}.$$

2.
$$ay'' + by' + cy = 0 \qquad (a \neq 0)$$

its characteristic equation:
$$ar^2 + br + c = 0.$$

3. **Method of Undetermined Coefficients**: If in the equation

$$ay'' + by' + cy = g(t), \qquad a \neq 0 \text{ és } t \in I$$

the right-hand side function $g(t)$ has the form

$$g(t) = \mathrm{e}^{ut} \left(A_n(t) \cos(vt) + B_m(t) \sin(vt) \right),$$

where $A_n(t), B_m(t)$ are polynomials of degree n and m respectively, then the particular solution of the inhomogeneous equation has the form:

$$y_{i,p} = t^s \mathrm{e}^{ut} \left(P_k(t) \cos(vt) + Q_k(t) \sin(vt) \right),$$

where s is the multiplicity of the root $u + i \cdot v$ among the roots of the characteristic equation; further, $P_k(t)$ and $Q_k(t)$ are polynomials of degree $k = \max(n, m)$.

4. **Variation of Parameters Method**: Consider the inhomogeneous d.e.

$$y'' + p(t)y' + q(t)y = g(t) \qquad t \in I$$

and its homogeneous part $Y'' + p(t)Y' + q(t)Y = 0$. If the y_1, y_2 pair is a fundamental solution of the homogeneous d.e., then a particular solution of the inhomogeneous equation is looked for in the form $y_{i,p} = C_1(t) \cdot y_1(t) + C_2(t) \cdot y_2(t)$, where for the derivatives of the unknown functions $C_1(t), C_2(t)$ the following system of equations holds:
$$C_1'(t)y_1(t) + C_2'(t)y_2(t) = 0$$
$$C_1'(t)y_1'(t) + C_2'(t)y_2'(t) = g(t)$$

5. **Special second order d.e.'s:**

If y is missing, then substitute $p(x) := y'(x)$.

If x is missing, then substitute $q(y) := y'$

6. The first order d.e. $M(x, y)dx + N(x, y)dy = 0$ is **exact**, if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

To solve the d.e., a function $F : \mathbb{R}^2 \to \mathbb{R}$ has to be found such that $\mathrm{grad} F = (M, N)$. Then the solution of the d.e. is:
$$F(x, y) = \text{Const}.$$