Problems for the first week

1. Find the general solution of the differential equation

$$y' = \frac{y}{x}.$$

2. Find the general solution of the differential equation

$$y' = \mathrm{e}^{x+y}.$$

3. Consider a sample of radioactive material which has mass $y(t)$ kg at time t. It has been observed that a constant factor of those radioactive atoms will spontaneously decay (into atoms of another element or into another isotope of the same element) during each time unit.

   (a) Find the differential equation which describes this process if the half-life of the material is $T = 100sec$.

   (b) Assume that at the beginning we had $1kg$ from this material. Find $y(t)$.

   The **half-life** of a radioactive material is the time for an amount of this material to decay to one-half of its original value.

4. Find the solution of the following initial value problem:

$$y' = \frac{\mathrm{e}^x}{y+1} \quad ; \quad y(0) = -4.$$

5. Assume that as a result of the drag force the decay of the speed of a moving object is proportional to the square of the speed of the object. Let $v(t)$ be the velocity as a function of the time. Write a differential equation for $v(t)$ and solve this differential equation.

6. Find the general solution of the differential equation

$$y' = \frac{1 + 2\mathrm{e}^y}{\mathrm{e}^y x \ln(x)}.$$

7. Find the general solution of the differential equation

$$(\mathrm{e}^{-2y} - \mathrm{e}^{-y})y' = \frac{\mathrm{e}^{x-y} + \mathrm{e}^{-x-y}}{\mathrm{e}^y + 1}.$$

8. Find the solution of the following initial value problem:

$$x + y - xy' = 0 \quad ; \quad y(1) = 1.$$

9. Find the general solution of the differential equation

$$x\mathrm{e}^{y/x} + y = xy'.$$

10. Find the general solution of the differential equation

$$xy' = y(\ln y - \ln x).$$

Results and some Solutions

1.) $y = C \cdot x$, where $C \neq 0$.
2.) $y = -\ln(-e^x - C)$, where $C < 0$.
3.)

(a) The differential equation is:

$$\frac{dy}{dt} = Ay.$$

Thus $y = Ce^{At}$. Since the half life is $T = 100$ we can write

$$y(100) = \frac{1}{2} \cdot y(0).$$

This yields $A = -\frac{\ln 2}{100}$. So the general solution is

$$y = Ce^{-\frac{\ln 2}{100} \cdot t}$$

(b) Using that $y(0) = 1$ we obtain that $C = 1$. Substitute this into the previous formulae to get

$$y(t) = e^{-\frac{\ln 2}{100} \cdot t}$$

4.) We write the equation in the form

$$\int (y+1)dy = \int e^x dx.$$

That is

$$\frac{y^2}{2} + y = e^x + C.$$

Solving this for y yields

$$y(x) = -1 \pm \sqrt{1 + 2(e^x + C)}.$$

Using that $y(0) = -4$ we obtain that $C = 3$. Thus the solution of the initial value problem is:

$$y(x) = -1 - \sqrt{7 + 2e^x}.$$

5.) The differential equation is:

$$-\frac{dv}{dt} = Av^2$$

for a constant $A < 0$. The solution of this equation is

$$v(t) = \frac{1}{At + C}.$$

6.) We write the separable equation in the form

$$\int \frac{e^y}{1 + 2e^y} dy = \int \frac{1}{x \ln x} dx.$$

Apply the substitution $u = e^y, v = \ln x$ to get

$$\int \frac{du}{1 + 2u} = \int \frac{dv}{v}.$$

After integrating both sides we obtain

$$\frac{1}{2}\ln(1+2u) = \ln(|v|) + C.$$

This yields

$$u = \frac{1}{2}\left(\mathrm{e}^{2C}v^2 - 1\right).$$

That is

$$y = \ln u = \ln\left(\mathrm{e}^{2C}\left(\ln x\right)^2 - 1\right) - \ln 2.$$

7.) $y = \cosh^{-1}\left(\sinh(x) + C\right)$, where $\sinh(x) + C \geq 1$.
8.) After the substitution $u = \frac{y}{x}$ we get a separable equation. The general solution is : $y = x\ln(|x|) + Cx$. Using that $y(1) = 1$ we obtain that $C = 1$. Thus the solution of the initial value problem is: $y = x\ln(|x|) + x$.
9.) After the substitution $u = \frac{y}{x}$ we get a separable equation. This leads to the general solution: $y = x\ln\left(C + \ln|x|\right)$.
10.) First we use the substitution $u = \frac{y}{x}$. We obtain that

$$\int \frac{du}{u\ln u - u} = \int \frac{dx}{x}.$$

Then assuming that $x, y > 0$ we write $z = \ln u$. This leads to

$$\int \frac{dz}{z-1} = \int \frac{dx}{x}.$$

After integration: $\ln(|z-1|) = \ln(x) + C$. In this way we get

$$|z-1| = \mathrm{e}^C x.$$

If $z > 1$ (that is $y > \mathrm{e}\cdot x$) then

$$y = ux = \mathrm{e}^z x = \mathrm{e}^{\mathrm{e}^C x + 1}x.$$

If $y < \mathrm{e}\cdot x$ then

$$y = ux = \mathrm{e}^z x = \mathrm{e}^{-\mathrm{e}^C x + 1}x.$$

During our calculations we excluded the case $y = \mathrm{e}\cdot x$ which is also a solution of the equation.