Problems for the third week

1. In each of the following problems (a) through (d) find the solution of the given initial value problem and compute $\lim_{t\to\infty} y(t)$.

   (a) $y'' + 5y' + 6y = 0, \qquad y(0) = 2, y'(0) = 3$
   (b) $y'' + y' - 2y = 0, \qquad y(0) = 1, y'(0) = 1,$
   (c) $y'' + 4y' + 3y = 0, \qquad y(0) = 2, y'(0) = -1,$
   (d) $y'' + 8y' - 9y = 0, \qquad y(1) = 1, y'(1) = 0.$

2. Consider the initial value problem

$$y'' + 5y' + 6y = 0, \qquad y(0) = 2, y'(0) = \beta,$$

   where $\beta > 0$.

   (a) Solve the initial value problem.
   (b) Determine the coordinates (t_m, y_m) of the maximum point of the solution as functions of β
   (c) Determine the smallest value of β for which $y_m \geq 4$.
   (d) Determine the behavior of t_m and y_m as $\beta \to \infty$.

3. In each problems (a) through (d) use Euler's formula to write the given expression in the form $a + ib$.

   (a) e^{-3+6i}
   (b) e^{1+2i}
   (c) $\mathrm{e}^{i\pi}$
   (d) 2^{1-i}

4. In each of the following problems (a) through (d) find the solution of the given initial value problem

   (a) $16y'' - 8y' + 145y = 0, \qquad y(0) = -2, y'(0) = 1$
   (b) $y'' + 4y = 0, \qquad y(0) = 1, y'(0) = 1,$
   (c) $y'' - 2y' + 5y = 0, \qquad y(\pi/2) = 0, y'(\pi/2) = 2,$

(d) $y'' + 2y' + 2y = 0, \qquad y(\pi/4) = 2, y'(\pi/4) = -2.$

5. In each of the following problems (a) through (d) find the solution of the given initial value problem

   (a) $y'' - y' + 0.25y = 0, \qquad y(0) = 2, y'(0) = \frac{1}{3}$

   (b) $9y'' - 12y' + 4y = 0, \qquad y(0) = 2, y'(0) = -1$

   (c) $9y'' + 6y' + 82y = 0, \qquad y(0) = -1, y'(0) = 2$

   (d) $y'' + 4y' + 4y = 0, \qquad y(-1) = 2, y'(-1) = 1$

6. Consider the initial value problem

$$4y'' + 12y' + 9y = 0, \qquad y(0) = 1, y'(0) = -4.$$

   (a) Solve the initial value problem and plot its solution for $0 \leq t \leq 5$.

   (b) Determine where the solution has the value zero.

   (c) Determine the coordinates (t_0, y_0) of the minimum points.

   (d) Change the second initial condition to $y'(0) = b$ and find the solution as a function of b. Then find the critical value of b that separates solutions that always remain positive from those that eventually become negative.

Results for all exercises and fully worked out solutions (for 1. (a), 3.(a), 4.(a), 5.(a))

1. (a) The characteristic equation is:

$$r^2 + 5r + 6 = 0.$$

The roots are: $r_1 = -2, r_2 = -3$. Thus the general solution is:

$$y = c_1 \mathrm{e}^{-2t} + c_2 \mathrm{e}^{-3t}. \tag{1}$$

The derivative of the general solution is:

$$y' = -2c_1 \mathrm{e}^{-2t} - 3c_2 \mathrm{e}^{-3t}. \tag{2}$$

Substitute the initial conditions into equations (1) and (2) to get

$$
\begin{aligned}
2 = y(0) &= c_1 + c_2 \\
3 = y'(0) &= -2c_1 - 3c_2
\end{aligned}
$$

Solving this we obtain:

$$c_1 = 9, \quad c_2 = -7$$

We substitute these into (1) and get the solution of the initial value problem:

$$y = 9\mathrm{e}^{-2t} - 7\mathrm{e}^{-3t}.$$

1. (b) $y = \mathrm{e}^t$, $\lim\limits_{t\to\infty} y(t) = \infty$,

1. (c) $y = \frac{5}{2}\mathrm{e}^{-t} - \frac{1}{2}\mathrm{e}^{-3t}$, $\lim\limits_{t\to\infty} y(t) = 0$.

1. (d) $y = \frac{1}{10}\mathrm{e}^{-9(t-1)} + \frac{9}{10}\mathrm{e}^{t-1}$, $\lim\limits_{t\to\infty} y(t) = \infty$.

2. (a) $y = (6+\beta)\mathrm{e}^{-2t} - (4+\beta)\mathrm{e}^{-3t}$,

2. (b) $t_m = \ln\left[(12+3\beta)\,/\,(12+2\beta)\right]$, $y_m \frac{4}{27}(6+\beta)^3/(4+\beta)^2$,

2. (c) $\beta = 6(1+\sqrt{3})$,

2. (d) $t_m \to \ln(3/2)$.

3. (a) $\mathrm{e}^{-3+6i} = \mathrm{e}^{-3}\mathrm{e}^{-6i} = \mathrm{e}^{-3}\left(\cos 6 + i\sin 6\right) \cong 0.0478 - 0.0139 \cdot i$.

3. (b) $\mathrm{e}\cos 2 + i\mathrm{e}\sin 2$,

3. (c) -1,

3. (d) $2\cos(\ln 2) - 2i\sin(\ln 2)$.

4. (a) The characteristic equation is:

$$16r^2 - 8r + 145 = 0$$

and its roots are: $r_1 = 1/4 + 3i, r_2 = 1/4 - 3i$. The general solution is

$$y = c_1\mathrm{e}^{t/4}\cos(3t) + c_2\mathrm{e}^{t/4}\sin(3t). \tag{3}$$

The derivative of the general solution is:

$$y' = 1/4\, c_1\, e^{1/4\,t}\cos(3\,t) - 3\, c_1\, e^{1/4\,t}\sin(3\,t) + 1/4\, c_2\, e^{1/4\,t}\sin(3\,t) + 3\, c_2\, e^{1/4\,t}\cos(3\,t) \tag{4}$$

Now we substitute the initial values into the equations (3) and (4) and get

$$
\begin{aligned}
-2 = y(0) &= c_1 \\
1 = y'(0) &= \frac{1}{4}c_1 + 3c_2.
\end{aligned}
$$

Thus we obtain $c_1 = -2$ and $c_2 = 1/2$. We substitute these into (3) to get the solution of the initial value problem:

$$y = -2\mathrm{e}^{t/4}\cos(3t) + \frac{1}{2}\mathrm{e}^{t/4}\sin(3t).$$

4. (b) $y = \frac{1}{2}\sin(2t)$
4. (c) $y = -\mathrm{e}^{t-\pi/2}\sin(2t)$
4. (d) $y = \sqrt{2}\mathrm{e}^{-(t-\pi/4)}\cos t + \sqrt{2}\mathrm{e}^{-(t-\pi/4)}\sin t$

5. (a) The characteristic equation is:

$$r^2 - r + 0.25 = 0$$

and the roots are $r := r_1 = r_2 = 1/2$. The general solution is:

$$y = c_1\mathrm{e}^{t/2} + c_2 \cdot t \cdot \mathrm{e}^{t/2}. \tag{5}$$

The derivative of the general solution is:

$$y' = 1/2 \cdot\ c_1\, e^{1/2\,t} + c_2\, e^{1/2\,t} + 1/2 \cdot\ c_2\, te^{1/2\,t}. \tag{6}$$

Substitute the initial values into equations (5) and (6). We get

$$\begin{aligned} 2 = y(0) &= c_1 \\ \frac{1}{3} = y'(0) &= \frac{1}{2}c_1 + c_2 \end{aligned}$$

Thus $c_1 = 2$ and $c_2 = -\frac{2}{3}$. We substitute these into (5) to get the solution of the initial value problem:

$$y = 2\mathrm{e}^{t/2} - \frac{2}{3} \cdot t \cdot \mathrm{e}^{t/2}.$$

5. (b) $y = 2\mathrm{e}^{2t/3} - \frac{7}{3}t\mathrm{e}^{2t/3}$,
5. (c) $y = -\mathrm{e}^{-t/3}\cos(3t) + \frac{5}{9}\mathrm{e}^{-t/3}\sin(3t)$,
5. (d) $y = 7\mathrm{e}^{-2(t+1)} + 5t\mathrm{e}^{-2(t+1)}$.

6. (a) $\mathrm{e}^{-3t/2} - \frac{5}{2}t\mathrm{e}^{-3t/2}$,
6. (b) $t = 2/5$,
6. (c) $t_0 = 16/15$, $y_0 = -\frac{5}{3}\mathrm{e}^{-8/5}$
6. (d) $y = \mathrm{e}^{-3t/2} + \left(b + \frac{3}{2}\right)t\mathrm{e}^{-3t/2}$, $b = -\frac{3}{2}$.