Problems for the fourth week with results and fully worked out solutions to Problems 1 (a)–(e),4,9

1. Use the method of undetermined coefficients to find the general solution of the following five differential equations:

   (a) $y'' - 3y' - 4y = 3e^{2t}$

   (b) $y'' - 3y' - 4y = 2\sin t$

   (c) $y'' - 3y' - 4y = -8t\cos(2t)$

   (d) $y'' - 3y' - 4y = 3e^{2t} + 2\sin t - 8t\cos(2t)$

   (e) $y'' - 3y' - 4y = 2e^{-t}$

2. In each of the following four problems find the general solution of the differential equation

   (a) $y'' - 2y' - 3y = 3e^{2t}$

   (b) $y'' + 2y' + 5y = 3\sin(2t)$

   (c) $y'' - 2y' - 3y = -3te^{-t}$

   (d) $y'' + 2y' + y = 2e^{-t}$

3. Find the solution of the initial value problem:

$$y'' + 4y = t^2 + 3t, \qquad y(0) = 0, y'(0) = 2.$$

4. Find the general solution of

$$y'' + 4y = 3\csc t, \qquad (\csc t = 1/\sin t)$$

5. First use the method of variation of parameters to find the general solution of the following two differential equations. Then use the method of undetermined coefficients to check your answers.

   (a) $y'' - 5y' + 6y = 2e^{t}$.

   (b) $4y'' - 4y' + y = 16e^{t/2}$.

6. Find the general solution of the differential equation

$$y'' + y = \tan t, \qquad 0 < t < \frac{\pi}{2}.$$

7. Find the general solution of the differential equation

$$4y'' + y = 2\sec(t/2), \qquad -\pi < t < \pi \ (\sec t = 1/\cos t).$$

8. Consider the

$$t^2y'' - t(t+2)y' + (t+2)y = 2t^3, \quad t > 0.$$

First verify that the functions

$$Y_1 = t, \qquad Y_2 = te^t$$

form a fundamental solution of the corresponding homogenous equation $t^2y'' - t(t+2)y' + (t+2)y = 0$. Then find the general solution of the inhomogeneous equation.

9. A mass 4 lb stretches a spring 2 in. Supposed that the mass is displaced an additional 6 in (1/2 ft) in the positive direction and then released. The mass is in a medium that exerts a viscous resistance of 6 lb when the mass has a velocity of 3 lb/sec. Formulate the initial value problem that governs the $y(t)$ motion of the mass.

Results

1.Since

$$y_{i,gen} = Y_{h,gen} + y_{i,p} \tag{1}$$

first we solve the homogenous part

$$Y'' - 3Y' - 4Y = 0.$$

of the equation. Since the roots of the characteristic equation $r^2 - 3r - 4$ are

$$r_1 = -1, \qquad r_2 = 4 \tag{2}$$

we obtain that the general solution of the homogenous part of the equation is:

$$Y_{h,alt} = c_1 e^{-t} + c_2 e^{4t}. \tag{3}$$

Observe that this is the same in all the five part of this problem. So, we only need to find a particular solution in each of the next five problems.

1a. Using that 2 is not a root of the characteristic equation we should try to find $y = y_{i,p}$ in the form

$$y = c \cdot \mathrm{e}^{2t}.$$

To find the constant we need to substitute back to the differential equation $y'' - 3y' - 4y = 3\mathrm{e}^{2t}$. For that we need

$$y' = 2c\mathrm{e}^{2t}, \qquad y'' = 4c\mathrm{e}^{2t}.$$

Thus

$$(4c - 3 \cdot 2c - 4 \cdot c) \cdot \mathrm{e}^{2t} = 3\mathrm{e}^{2t}.$$

This yields that $c = -1/2$. That is

$$y = y_{i,p} = -\frac{1}{2}\mathrm{e}^{2t}.$$

This and (3) together implies that

$$y_{i,alt} = c_1\mathrm{e}^{-t} + c_2\mathrm{e}^{4t} - \frac{1}{2}\mathrm{e}^{2t}$$

1b. We only need to find $y = y_{i,p}$. We try to find it in the form:

$$y = A\cos t + B\sin t.$$

That is we need to compute A and B such that $y = A\cos t + B\sin t$ is a solution of

$$y'' - 3y' - 4y = 2\sin t. \tag{4}$$

Thus we need

$$y' = -A\sin t + B\cos t, \qquad y'' = -A\cos t - B\sin t.$$

After substituting this to (4) we obtain

$$(-5A - 3B)\cos t + (3A - 5B)\sin t = 2\sin t.$$

The coefficients of $\cos t$ and the coefficients of $\sin t$ must be the same:

$$\begin{aligned} -5A - 3B &= 0 \\ 3A - 5B &= 2 \end{aligned}$$

So, $A = 3/17$ and $B = -5/17$ therefore

$$y_{i,p} = \frac{3}{17}\cos t - \frac{5}{17}\sin t.$$

Using this, (1) and (3) we obtain

$$y_{i,alt} = \underbrace{c_1 \mathrm{e}^{-t} + c_2 \mathrm{e}^{4t}}_{Y_{h,alt}} + \underbrace{\frac{3}{17}\cos t - \frac{5}{17}\sin t}_{y_{i,p}}.$$

1c. We need to find A, B such that the function $y_{i,p} = A\mathrm{e}^t \cos(2t) + B\mathrm{e}^t \sin(2t)$ is a solution of $y'' - 3y' - 4y = -8t\cos(2t)$. After differentiating twice:

$$y' = (A + 2B)\,\mathrm{e}^t \cos(2t) + (-2A + B)\,\mathrm{e}^t \sin(2t),$$
$$y'' = (-3A + 4B)\,\mathrm{e}^t \cos(2t) + (-4A - 3B)\,\mathrm{e}^t \sin(2t).$$

Substituting into $y'' - 3y' - 4y = -8t\cos(2t)$ yields

$$\begin{aligned} 10A + 2B &= 8 \\ 2A - 10B &= 0. \end{aligned}$$

The solution is $A = 10/13$ and $B = 2/13$. Thus

$$y_{i,p} = \frac{10}{13}\mathrm{e}^t \cos(2t) + \frac{2}{13}\mathrm{e}^t \sin(2t).$$

Using this, (1) and (3) we obtain

$$y_{i,alt} = \underbrace{c_1 \mathrm{e}^{-t} + c_2 \mathrm{e}^{4t}}_{Y_{h,alt}} + \underbrace{\frac{10}{13}\mathrm{e}^t \cos(2t) + \frac{2}{13}\mathrm{e}^t \sin(2t)}_{y_{i,p}}.$$

1d. Observe that the right hand side of the equation is just the sum of the right hand sides of the previous three equations and the left hand side is the same. Using that the equations are linear it follows that the we get a particular solution as the sum of the particular solutions in the previous three problems. That is

$$y_{i,p} = -\frac{1}{2}\mathrm{e}^{2t} + \frac{3}{17}\cos t - \frac{5}{17}\sin t + \frac{10}{13}\mathrm{e}^t \cos(2t) + \frac{2}{13}\mathrm{e}^t \sin(2t)$$

Thus the general solution is:

$$y_{i,alt} = \underbrace{c_1\mathrm{e}^{-t} + c_2\mathrm{e}^{4t}}_{Y_{h,alt}} + \underbrace{-\frac{1}{2}\mathrm{e}^{2t} + \frac{3}{17}\cos t - \frac{5}{17}\sin t + \frac{10}{13}\mathrm{e}^{t}\cos(2t) + \frac{2}{13}\mathrm{e}^{t}\sin(2t)}_{y_{i,p}}.$$

1e. Using (2) we see that -1 is a simple root of the characteristic polynomial. Thus we try to find particular solution in the form

$$y = (At + B)\mathrm{e}^{-t}.$$

To do so, we need to find the constants A and B for which $y = (At + B)\mathrm{e}^{-t}$ is a solution of $y'' - 3y' - 4y = 2\mathrm{e}^{-t}$. First we compute

$$y' = (A - B)e^{-t} - Ate^{-t}, \qquad y'' = (-2\,A + B)e^{-t} + Ate^{-t}.$$

After substitution we obtain a solution $A = -2/3$, $B = 0$. Thus a particular solution of $y'' - 3y' - 4y = 2\mathrm{e}^{-t}$ is

$$y_{i,p} = -\frac{2}{3}t\mathrm{e}^{-t}.$$

Using this, (1) and (3) we obtain

$$y_{i,gen} = \underbrace{c_1\mathrm{e}^{-t} + c_2\mathrm{e}^{4t}}_{Y_{h,gen}} + \underbrace{-\frac{2}{3}t\mathrm{e}^{-t}}_{y_{i,p}}.$$

2a. $y = c_1\mathrm{e}^{3t} + c_2\mathrm{e}^{-t} - \mathrm{e}^{2t}$
2b. $y = c_1\mathrm{e}^{-t}\cos(2t) + c_2\mathrm{e}^{-t}\sin(2t) + \frac{3}{17}\sin(2t) - \frac{12}{17}\cos(2t)$
2c.$y = c_1\mathrm{e}^{3t} + c_2\mathrm{e}^{-t}\frac{3}{16}t\mathrm{e}^{-t} + \frac{3}{8}t^2\mathrm{e}^{-t}$
2d.$y = c_1\mathrm{e}^{-t} + c_2t\mathrm{e}^{-t} + t^2\mathrm{e}^{-t}$.
3. $y = \frac{7}{10}\sin(2t) - \frac{19}{40}\cos(2t) + \frac{1}{4}t^2 - \frac{1}{8} + \frac{3}{5}\mathrm{e}^{t}$
4. The general solution is given by

$$y_{i,gen} = Y_{h,gen} + y_{i,p} \tag{5}$$

The homogenous part is

$$Y'' + 4Y = 0.$$

The characteristic polynomial of this equation is $r^2 + 4r = 0$. Therefore the roots of the characteristic polynomial are

$$r_1 = 2i, \qquad r_2 = -2i.$$

the general solution of the homogenous part is

$$Y_{h,alt} = c_1 \cos(2t) + c_2 \sin(2t). \tag{6}$$

To find $y = y_{i,p}$ we need to use the Variation of parameters method: That is we need to determine the functions $c_1(t), c_2(t)$ such that

$$y = c_1(t) \cos(2t) + c_2(t) \sin(2t) \tag{7}$$

is a solution of $y'' + 4y = 3 \csc t$. For that we need to solve the system of equations

$$\begin{aligned} c_1'(t) \cos(2t) + c_2'(t) \sin(2t) &= 0 \\ -2c_1'(t) \sin(2t) + 2c_2'(t) \cos(2t) &= 3 \csc t. \end{aligned}$$

From the first equation we get

$$c_2'(t) = -c_1'(t) \frac{\cos(2t)}{\sin(2t)}.$$

Then substituting this into the second equation yields:

$$c_1'(t) = -\frac{3 \csc t \sin(2t)}{2} = -3 \cos t.$$

Substituting this back to the one but last equation we get

$$c_2'(t) = \frac{3}{2} \csc t - 3 \sin t.$$

Now we integrate (without the additive constants since we need only one particular solution) and we get

$$c_1(t) = -3 \sin(t), \;\; c_2(t) = \frac{3}{2} \ln |\csc t - \cot t| + 3 \cos t.$$

Thus

$$y = y_{i,p} = (-3 \sin(t)) \cdot \cos(2t) + (\frac{3}{2} \ln |\csc t - \cot t| + 3 \cos t) \cdot \sin(2t).$$

Using (5) and (6) the general solution is

$$\begin{aligned} y_{i,gen} &= c_1 \cos(2t) + c_2 \sin(2t) \\ &+ (-3 \sin(t)) \cdot \cos(2t) + (\frac{3}{2} \ln |\csc t - \cot t| + 3 \cos t) \cdot \sin(2t). \end{aligned}$$

5a. $y = c_1 e^{2t} + c_2 e^{3t} + e^t$.
5b. $y = c_1 e^{t/2} + c_2 t e^{t/2} + 2t^2 e^{t/2}$.
6. $y = c_1 \cos t + c_2 \sin t - (\cos t) \ln(\tan t + \sec t)$.
7. $y = c_1 \cos(t/2) + c_2 \sin(t/2) + t \sin(t/2) + 2 [\ln \cos(t/2)] \cos(t/2)$.
8. $y = c_1 t + c_2 t e^t - 2t^2$.
9. First we want to write down the equation:

$$my''(t) + \gamma y'(t) + ky(t) = F(t),$$

We measure the displacement in ft the mass in lb and the time in sec. We recall that the acceleration due to the gravity is $g = 32\frac{ft}{sec^2}$. First we observe that there is no external force. So $F(t) = 0$. We determine the mass m from

$$mg = 4\ lb.$$

That is

$$m = \frac{1}{8} \frac{lb \cdot sec^2}{ft}.$$

We determine the damping coefficient γ from the assumption that the damping force $\gamma y' = 6lb$ when $y' = 3ft/sec$. Thus

$$\gamma = \frac{6\ lb}{3\ ft/sec} = 2\frac{lb \cdot sec}{ft}.$$

The spring constant k is determined from the assumption that the mass stretches the spring by $L = 2in = 1/6ft$. Thus $4lb = m \cdot g = k \cdot L$ yields

$$k = \frac{4\ lb}{1/6ft} = 24\frac{lb}{ft}.$$

Thus the equation which governs the motion of the motion is:

$$y'' + 16y' + 192y = 0, \qquad y(0) = \frac{1}{2}, y'(0) = 0.$$