Problems and results for the fifth week with fully worked out solutions for the problems: 1,2,3.(a),(b),(c), 6.(a)

1. Solve the following initial value problem:

$$y'' = \frac{1}{\sqrt{1-x^2}}, \qquad y(0) = 3, \ y'(0) = 1$$

2. A rod is loaded by a bending moment that is proportinal to the value $f(x)$ at each coordinate x. It is known that the shape of this rod's median can be computed by solving the following differential equation:

$$\frac{y''}{(1+(y')^2)^{3/2}} = f(x).$$

Determine the shape of the rod if the bending moment follows

$$f(x) = 1 - x$$

and the initial conditions are given by

$$y(0) = y'(0) = 0.$$

3. (a) Consider the differential equation of free mechanical vibration without damping

$$my'' + ky = 0. \qquad (1)$$

Solve it as an incomplete second order differential equation.

(b) Solve differential equation (1) as a second order linear equation.

(c) Prove that the solutions you obtained in 3a and 3b are the same.

4. Find the general solution of the following differential equations.

(a)

$$(y')^2 + 2yy'' = 0,$$

(b)

$$y'' = \frac{1}{4\sqrt{y}},$$

(c)

$$yy'' + (y')^2 = 1$$

5. Solve the following second order differential equation:

$$xy'' - y' = x^3.$$

6. Solve the following differential equations:

   (a) $2x\cos y + [2y\cos y - (x^2 + y^2)\sin y]\, y' = 0,$

   (b) $xdy + ydx = 0,$

   (c) $\frac{x}{x^2+y^2} y' = \frac{y}{x^2+y^2},$

   (d) $2x(\sin y + 1) + x^2 \cos y \cdot y' = 0.$

Results

1. Integrate both sides twice against x:

$$y' = \int \frac{1}{\sqrt{1-x^2}} dx = \arcsin x + C_1,$$

$$y = \int (\arcsin x + C_1) dx = \arcsin x + \sqrt{1-x^2} + C_1 x + C_2.$$

Substitute into the initial conditions

$$\begin{aligned} 3 = y(0) &= 1 + C_2 \\ 1 = y'(0) &= \arcsin 0 + 1. \end{aligned}$$

This yields

$$C_1 = 1 \text{ and } C_2 = 2.$$

The solution of the initial value problem is:

$$y = \arcsin x + \sqrt{1-x^2} + x + 2.$$

2. This equation does not contain y. So, we introduce a new variable $p = y'$. Then

$$p(x) = y'(x) \text{ and } p'(x) = y''(x).$$

We substitute this into our equation and get

$$\int \frac{dp}{(1+p^2)^{3/2}} = \int f(x)dx.$$

Let us write

$$\int f(x)dx = F(x).$$

After integration

$$\frac{p}{\sqrt{1+p^2}} = F(x) + c_1.$$

Then it follows that

$$p(x) = \frac{F(x)+c_1}{\sqrt{1-(F(x)+C-1)^2}}.$$

Using that $y' = p$ yields

$$y = \int \frac{F(x)+c_1}{\sqrt{1-(F(x)+c_1)^2}} dx. \tag{2}$$

In the special case when $f(x) = 1 - x$ we get by integration that

$$F(x) + c_1 = x(1-\frac{x}{2}) + c_1.$$

That is

$$y' = p = \frac{x(1-\frac{x}{2})+c_1}{\sqrt{1-\left(x(1-\frac{x}{2})+c_1\right)^2}}.$$

Using the initial value $y'(0) = 0$ we get that

$$c_1 = 0.$$

Now we substitute this into (**??**). Using the other initial value $y(0) = 0$ follows that

$$y(x) = \int_{t=0}^{x} \frac{t(1-\frac{t}{2})}{\sqrt{1-t^2\left(1-\frac{t}{2}\right)^2}} dt.$$

This is an *elliptic integral* so we cannot express as a formula which contains elementary functions. However, we can draw its graph using computer (see Figure **??**).
3a. 1 Using the notation

$$\omega_0 := \sqrt{\frac{k}{m}}$$

our differential equation is

$$y'' = -\omega_0^2 y. \tag{3}$$

Since it does not contain the independent variable x explicitly we can introduce the new variable p as

$$y' = p = p(y), \qquad y'' = \frac{dp}{dy} p,$$

Using this substitution equation (**??**) is:

$$p \cdot p' = -\omega_0^2 y \tag{4}$$

Here we mean $p' = \frac{dp}{dy}$. Then equation (**??**) is a separable differential equation. We multiply by dy which follows

$$p \cdot dp = -\omega_0^2 y \cdot dy.$$

Interating both sides yields

$$\frac{1}{2} p^2 = -\frac{\omega_0^2}{2} y^2 + C_1.$$

That is

$$\frac{dy}{dt} = y' = p = \pm\sqrt{2C_1 - \omega_0^2 y^2}.$$

In this way we obtained the separable differential equation

$$\frac{dy}{dt} = \pm\sqrt{2C_1 - \omega_0^2 y^2}, \tag{5}$$

which is equivalent to

$$\frac{dy}{\sqrt{2C_1 - \omega_0^2 y^2}} = \pm dt.$$

Using that

$$\int \frac{1}{\sqrt{1-x^2}}dx = \arcsin x + C$$

we get

$$y = \frac{\sqrt{2C_1}}{\omega_0}\sin(\omega_0(t+C_2)). \tag{6}$$

In the one but last equation both the $+$ and $-$ signs results the same solution. To see this note that if the constants are C_2 and C_2' appropriately then $\omega_0 C_2' = \omega_0 C_2 + \pi$. But this is not important.

3b. Consider equation (1) as a linear equation. Its characteristics polynomial is

$$mr^2 - k = 0.$$

Using the notation $\omega_0 := k/m$ we obtain that the roots are:

$$r_1 = i \cdot \omega_0, \qquad r_2 = i \cdot \omega_0.$$

The general solution is:

$$y = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t). \tag{7}$$

3c. In equation (**??**) on the right hand side we pull out $\sqrt{c_1^2+c_2^2}$-et. Then we get

$$y = \sqrt{c_1^2+c_2^2} \cdot \left(\frac{c_1}{\sqrt{c_1^2+c_2^2}} \cos(\omega_0 t) + \frac{c_2}{\sqrt{c_1^2+c_2^2}} \sin(\omega_0 t) \right). \tag{8}$$

Note that the vector

$$\mathbf{v} = \left(\frac{c_2}{\sqrt{c_1^2+c_2^2}}, \frac{c_1}{\sqrt{c_1^2+c_2^2}} \right)$$

is a unit vector. Thus there is an angle θ such that $\mathbf{v} = (\cos\theta, \sin\theta)$. For this angle θ we have

$$\cos\theta = \frac{c_2}{\sqrt{c_1^2+c_2^2}} \text{ and } \sin\theta = \frac{c_1}{\sqrt{c_1^2+c_2^2}}.$$

Substituting this into equation (**??**) yields:

$$y = \sqrt{c_1^2+c_2^2} \cdot \underbrace{(\sin\theta \cdot \cos(\omega_0 t) + \cos\theta \cdot \sin(\omega_0 t))}_{\sin(\theta + \omega_0 t)}.$$

Using the notation $d_1 :=\sqrt{c_1^2 + c_2^2}$ and $d_2 := \theta/\omega_0$ we get

$$y = d_1 \sin(\omega_0(t + d_2)). \tag{9}$$

This gives the same as (**??**) with $d_2 = c_2$ and $d_1 = \frac{\sqrt{2C_1}}{\omega_0}$.

4a. $y = C_1(x + C_2)^{2/3}$,

4b. $3x = 4(\sqrt{y} - 2C - 1)\sqrt{C_1 + \sqrt{y}} + C_2$,

4c. $(x + C_2)^2 - y^2 = C_1$.

5. $y = \frac{x^4}{8} + \frac{C_1 x^2}{2} + C - 2$.

6a. We write $y' = \frac{dy}{dx}$ and multiply both sides with dx.

$$\underbrace{2x\cos y}_{M(x,y)}\,dx + \underbrace{\left[2y\cos y - (x^2 + y^2)\sin y\right]}_{N(x,y)}\,dy = 0$$

Using that

$$\frac{\partial M}{\partial y} \equiv \frac{\partial N}{\partial x} = -2x\sin y$$

we see that the differential equation is exact. That is there exist functions $F : \mathbb{R}^2 \to \mathbb{R}$ such that

$$\mathbf{grad}(F) = (M, N).$$

That is

$$\frac{\partial F}{\partial x} = M, \qquad \frac{\partial F}{\partial y} = N.$$

Using that $\frac{\partial F}{\partial x} = M$ we can write

$$F(x, y) = \int M(x, y)dx + h(y) = x^2\cos y + h(y). \tag{10}$$

We obtain the function $h(y)$ from $\frac{\partial F}{\partial y} = N$. Namely,

$$\frac{\partial F}{\partial y} = -x^2\sin y + h'(y) = \underbrace{2y\cos y - (x^2 + y^2)\sin y}_{N}.$$

This yields

$$h'(y) = 2y\cos y - y^2\sin y.$$

After integration

$$h(y) = y^2\cos y + C_1.$$

We substitute this back into (**??**). In this way we get that

$$F(x, y) = (x^2 + y^2)\cos y + C_1.$$

So, the general solution of the differential equation is

$$(x^2 + y^2)\cos y = Const.$$

6b. $xy = Const$

6c. $\arctan \frac{y}{x} = Const.$

6d. $x^2 \sin y + \sin y = Const.$