**Problems and results for the seventh week**
Mathematics A3 for Civil Engineering students

1. Solve the following system of differential equations:
$$\begin{aligned} x' + 2y' - 3x + 4y &= 2\sin t \\ 2x' + y' + 2x - y &= \cos t. \end{aligned}$$

2. Solve the following system of differential equations:
$$\begin{aligned} x' - 2x + y &= \mathrm{e}^t \\ y' - 3x + 2y &= t. \end{aligned}$$

3. Solve the following system of differential equations:
$$\begin{aligned} x' - 2x + 5y &= -\cos t \\ y' - x + 2y &= \sin t. \end{aligned}$$

4. Solve the following system of differential equations:
$$\begin{aligned} x' - x - y &= \mathrm{e}^{-2t} \\ y' - 4x + 2y &= -2\mathrm{e}^t. \end{aligned}$$

5. Suppose that $A$ and $B$ are mutually exclusive events for which $\mathbb{P}\{A\} = 0.3$ and $\mathbb{P}\{B\} = 0.5$. What is the probability that
   (a) either $A$ or $B$ occurs;
   (b) $A$ occurs but $B$ does not;
   (c) both $A$ and $B$ occur?

6. A total of 28% of males smoke cigarettes, 7% smoke cigars, and 5% smoke both cigarettes and cigars.
   (a) What percentage of males smoke neither cigars nor cigarettes?
   (b) What percentage smoke cigars but not cigarettes?

7. A small community organization consists of 20 families, of which 4 have one child, 8 have two children, 5 have three children, 2 have four children, and 1 has five children.
   (a) If one of these families is chosen at random, what is the probability it has $i$ children, $i = 1, 2, 3, 4, 5$?
   (b) If one of the children is randomly chosen, what is the probability this child comes from a family having $i$ children, $i = 1, 2, 3, 4, 5$?

8. We flip a fair coin twice. What is the probability that at least one of our flips comes out Heads? What is the probability that *precisely* one of our flips comes out Heads?

9. What is the probability that, when rolling two fair dice, at least one of them shows a six? What is the probability that none of them show a six?

10. A pair of fair dice are rolled. What is the probability that the second die lands on a higher value than does the first?

11. Two symmetric dice have both had two of their sides painted red, two painted black, one painted yellow, and the other painted white. When this pair of dice are flipped, what is the probability that both land on the same color?

12. What is the probability that all three children of a family have the same sex? (We assume that each child is born girl or boy independently with probability 1/2.)

13. At least how many coin flips are needed in order to see at least one Heads with probability at least 90%?

14. We flip a fair die six times. What is the probability that each of the six numbers shows up?

15. Five people, designated as $A$, $B$, $C$, $D$, $E$, are arranged in linear order. Assuming that each possible order is equally likely, what is the probability that
    (a) there is exactly one person between $A$ and $B$;
    (b) there are exactly two people between $A$ and $B$;
    (c) there are exactly three people between $A$ and $B$?

16. What is the probability that in a classroom of 30 students there are no two birthdays that coincide? (We assume that each birthday is independently equally likely to be on any of the 365 days of the year. Do not bother with leap years.)

17. What is the probability that we will precisely have two hits on the lottery, where 5 numbers are drawn out of 90?

18. A forest contains 20 deers, of which 5 are captured, tagged, and then released. A certain time later 4 of the 20 deers are captured. What is the probability that 2 of these 4 have been tagged? What assumptions are you making?

19. There are 30 psychiatrists and 24 psychologists attending a certain conference. Three of these 54 people are randomly chosen to take part in a panel discussion. What is the probability that at least one psychologist is chosen?

## Answers

1. Subtract the second equation twice from the first one:

$$\begin{aligned}-3x' - 7x + 6y &= 2\sin t - 2\cos t,\\ x' &= -\frac{7}{3}x + 2y - \frac{2}{3}\sin t + \frac{2}{3}\cos t.\end{aligned}$$

Then subtract the first equation twice from the second one:

$$\begin{aligned}-3y' + 8x - 9y &= -4\sin t + \cos t,\\ y' &= \frac{8}{3}x - 3y + \frac{4}{3}\sin t - \frac{1}{3}\cos t.\end{aligned}$$

The above two displays are combined to

$$\underline{x}' = \begin{pmatrix} -\frac{7}{3} & 2 \\ \frac{8}{3} & -3 \end{pmatrix} \cdot \underline{x} + \begin{pmatrix} -\frac{2}{3}\sin t + \frac{2}{3}\cos t \\ \frac{4}{3}\sin t - \frac{1}{3}\cos t \end{pmatrix}. \tag{1}$$

To solve the homogeneous part, we find the eigenvalues as

$$0 = \begin{vmatrix} -\frac{7}{3} - r & 2 \\ \frac{8}{3} & -3 - r \end{vmatrix} = \Big(-\frac{7}{3} - r\Big) \cdot \Big(-3 - r\Big) - 2 \cdot \frac{8}{3} = r^2 + \frac{16}{3}r + \frac{5}{3},$$

from which $r^{(1)} = -1/3$ and $r^{(2)} = -5$. The corresponding eigenvalues come from

$$\begin{pmatrix} -\frac{7}{3} & 2 \\ \frac{8}{3} & -3 \end{pmatrix} \cdot \begin{pmatrix} v_1^{(1)} \\ v_2^{(1)} \end{pmatrix} = -\frac{1}{3} \cdot \begin{pmatrix} v_1^{(1)} \\ v_2^{(1)} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -\frac{7}{3} & 2 \\ \frac{8}{3} & -3 \end{pmatrix} \cdot \begin{pmatrix} v_1^{(2)} \\ v_2^{(2)} \end{pmatrix} = -5 \cdot \begin{pmatrix} v_1^{(2)} \\ v_2^{(2)} \end{pmatrix},$$

that is $\underline{v}^{(1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, and $\underline{v}^{(2)} = \begin{pmatrix} 3 \\ -4 \end{pmatrix}$. The general solution of the homogeneous system is thus

$$\underline{x}_{\text{homo.}} = c_1 \cdot \mathrm{e}^{-t/3} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \cdot \mathrm{e}^{-5t} \cdot \begin{pmatrix} 3 \\ -4 \end{pmatrix}.$$

To find the inhomogeneous particular solution we try

$$\underline{x}_{\text{inhomo.}} = c_1(t) \cdot \mathrm{e}^{-t/3} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2(t) \cdot \mathrm{e}^{-5t} \cdot \begin{pmatrix} 3 \\ -4 \end{pmatrix}.$$

Differentiating and plugging back to (1) gives

$$\begin{aligned} c_1'(t) \cdot \mathrm{e}^{-t/3} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2'(t) \cdot \mathrm{e}^{-5t} \cdot \begin{pmatrix} 3 \\ -4 \end{pmatrix} &= \begin{pmatrix} -\frac{2}{3}\sin t + \frac{2}{3}\cos t \\ \frac{4}{3}\sin t - \frac{1}{3}\cos t \end{pmatrix}, \text{ or}\\ \mathrm{e}^{-t/3}c_1'(t) + 3\mathrm{e}^{-5t}c_2'(t) &= -\frac{2}{3}\sin t + \frac{2}{3}\cos t,\\ \mathrm{e}^{-t/3}c_1'(t) - 4\mathrm{e}^{-5t}c_2'(t) &= \frac{4}{3}\sin t - \frac{1}{3}\cos t. \end{aligned}$$

Adding 3/4 times the second equation to the first one gives

$$\begin{aligned} \frac{7}{4}\mathrm{e}^{-t/3}c_1'(t) &= \frac{1}{3}\sin t + \frac{5}{12}\cos t,\\ c_1'(t) &= \frac{4}{21}\mathrm{e}^{t/3} \cdot \sin t + \frac{5}{21}\mathrm{e}^{t/3} \cdot \cos t,\\ c_1(t) &= \frac{19}{70}\mathrm{e}^{t/3} \cdot \sin t - \frac{1}{10}\mathrm{e}^{t/3} \cdot \cos t. \end{aligned}$$

Subtracting now the second equation from the first one leads to

$$\begin{aligned}
7\mathrm{e}^{-5t}c_2'(t) &= -2\sin t + \cos t,\\
c_2'(t) &= -\frac{2}{7}\mathrm{e}^{5t}\sin t + \frac{1}{7}\mathrm{e}^{5t}\cos t,\\
c_2(t) &= -\frac{9}{182}\mathrm{e}^{5t}\sin t + \frac{1}{26}\mathrm{e}^{5t}\cos t.
\end{aligned}$$

We can now write the general solution as

$$\begin{aligned}
\underline{x} = \underline{x}_{\text{homo.}} + \underline{x}_{\text{inhomo.}} &= c_1\cdot\mathrm{e}^{-t/3}\cdot\begin{pmatrix}1\\1\end{pmatrix} + c_2\cdot\mathrm{e}^{-5t}\cdot\begin{pmatrix}3\\-4\end{pmatrix}\\
&\quad + \left(\frac{19}{70}\sin t - \frac{1}{10}\cos t\right)\cdot\begin{pmatrix}1\\1\end{pmatrix} + \left(-\frac{9}{182}\sin t + \frac{1}{26}\cos t\right)\cdot\begin{pmatrix}3\\-4\end{pmatrix}\\
&= \begin{pmatrix} c_1\cdot\mathrm{e}^{-t/3} + 3c_2\cdot\mathrm{e}^{-5t} + \frac{8}{65}\sin t + \frac{1}{65}\cos t\\ c_1\cdot\mathrm{e}^{-t/3} - 4c_2\cdot\mathrm{e}^{-5t} + \frac{61}{130}\sin t - \frac{33}{130}\cos t\end{pmatrix}, \text{ or}\\
x &= c_1\cdot\mathrm{e}^{-t/3} + 3c_2\cdot\mathrm{e}^{-5t} + \frac{8}{65}\sin t + \frac{1}{65}\cos t,\\
y &= c_1\cdot\mathrm{e}^{-t/3} - 4c_2\cdot\mathrm{e}^{-5t} + \frac{61}{130}\sin t - \frac{33}{130}\cos t.
\end{aligned}$$

2. $x = c_1\mathrm{e}^{t} + c_2\mathrm{e}^{-t} + \frac{3}{2}t\mathrm{e}^{t} - \frac{1}{4}\mathrm{e}^{t} + t, \qquad y = c_1\mathrm{e}^{t} + 3c_2\mathrm{e}^{-t} + \frac{3}{2}t\mathrm{e}^{t} - \frac{3}{4}\mathrm{e}^{t} + 2t - 1.$

3. The system is rewritten as

$$\underline{x}' = \begin{pmatrix}2 & -5\\ 1 & -2\end{pmatrix}\cdot\underline{x} + \begin{pmatrix}-\cos t\\ \sin t\end{pmatrix}. \tag{2}$$

To solve the homogeneous system, we find the eigenvalues $r = \pm i$ and the corresponding eigenvectors

$$\underline{v} = \begin{pmatrix}5\\ 2\mp i\end{pmatrix} = \begin{pmatrix}5\\2\end{pmatrix} \pm i\cdot\begin{pmatrix}0\\-1\end{pmatrix}.$$

Therefore the solution of the homogeneous equation is

$$\begin{aligned}
\underline{x}_{\text{homo.}} &= c_1\left[\begin{pmatrix}5\\2\end{pmatrix}\cdot\cos t - \begin{pmatrix}0\\-1\end{pmatrix}\cdot\sin t\right] + c_2\left[\begin{pmatrix}5\\2\end{pmatrix}\cdot\sin t + \begin{pmatrix}0\\-1\end{pmatrix}\cdot\cos t\right]\\
&= \begin{pmatrix}5c_1\cos t + 5c_2\sin t\\ [2c_1 - c_2]\cos t + [c_1 + 2c_2]\sin t\end{pmatrix}.
\end{aligned}$$

We try the inhomogeneous solution in the form

$$\underline{x}_{\text{inhomo.}} = \begin{pmatrix}5c_1(t)\cos t + 5c_2(t)\sin t\\ [2c_1(t) - c_2(t)]\cos t + [c_1(t) + 2c_2(t)]\sin t\end{pmatrix}. \tag{3}$$

Plugging this in (2) gives

$$\begin{pmatrix}5c_1'(t)\cos t + 5c_2'(t)\sin t\\ [2c_1'(t) - c_2'(t)]\cos t + [c_1'(t) + 2c_2'(t)]\sin t\end{pmatrix} = \begin{pmatrix}-\cos t\\ \sin t\end{pmatrix}, \text{ or}$$

$$5c_1'(t)\cos t + 5c_2'(t)\sin t = -\cos t, \tag{4}$$

$$[2c_1'(t) - c_2'(t)]\cos t + [c_1'(t) + 2c_2'(t)]\sin t = \sin t. \tag{5}$$

From (4) we have $c_1'(t) = -c_2'(t)\tan t - 1/5$, which we plug in (5) to get

$$
\begin{aligned}
-2c_2'(t)\sin t - \frac{2}{5}\cos t - c_2'(t)\cos t - c_2'(t)\tan t\sin t - \frac{1}{5}\sin t + 2c_2'(t)\sin t &= \sin t,\\
c_2'(t)[\cos t + \tan t\sin t] &= -\frac{2}{5}\cos t - \frac{6}{5}\sin t,\\
c_2'(t) &= -\frac{2}{5}\cos^2 t - \frac{6}{5}\sin t\cos t, \qquad (6)\\
c_2(t) &= -\frac{1}{5}t - \frac{1}{5}\sin t\cos t - \frac{3}{10}\sin^2 t + \frac{3}{10}\cos^2 t.
\end{aligned}
$$

Turning back to $c_1(t)$, we have, from (6),

$$
\begin{aligned}
c_1'(t) &= -c_2'(t)\tan t - \frac{1}{5} = \frac{2}{5}\sin t\cos t + \frac{6}{5}\sin^2 t - \frac{1}{5},\\
c_1(t) &= \frac{1}{10}\sin^2 t - \frac{1}{10}\cos^2 t + \frac{2}{5}t - \frac{3}{5}\sin t\cos t.
\end{aligned}
$$

Plug these back in (3) to conclude

$$
\begin{aligned}
x &= 5c_1\cos t + 5c_2\sin t - \frac{1}{2}\cos t - \frac{3}{2}\sin t + 2t\cos t - t\sin t,\\
y &= [2c_1 - c_2]\cos t + [c_1 + 2c_2]\sin t - \frac{1}{2}\cos t - \frac{1}{2}\sin t + t\cos t.
\end{aligned}
$$

4. $x = c_1 e^{-3t} + c_2 e^{2t} + \frac{1}{2}e^t, \qquad y = -4c_1 e^{-3t} + c_2 e^{2t} - e^{-2t}.$

5.(a) $\mathbb{P}\{A \cup B\} = 0.3 + 0.5 = 0.8.$

(b) $\mathbb{P}\{A \cap B^c\} = \mathbb{P}\{A\} = 0.3.$

(c) $\mathbb{P}\{A \cap B\} = \mathbb{P}\{\emptyset\} = 0.$

6. Let $E$ be the event that a randomly selected male smokes cigarettes, $F$ the event that a randomly selected male smokes cigars.

(a) $\mathbb{P}\{E^c \cap F^c\} = 1 - \mathbb{P}\{E \cup F\} = 1 - \mathbb{P}\{E\} - \mathbb{P}\{F\} + \mathbb{P}\{E \cap F\} = 1 - 0.28 - 0.07 + 0.05 = 0.7$ or 70%.

(b) $\mathbb{P}\{F \cap E^c\} = \mathbb{P}\{F - E\} = \mathbb{P}\{F\} - \mathbb{P}\{F \cap E\} = 0.07 - 0.05 = 0.02$ or 2%.

7.(a) The probabilities are $\frac{4}{20}$, $\frac{8}{20}$, $\frac{5}{20}$, $\frac{2}{20}$, $\frac{1}{20}$, respectively for $i = 1, 2, 3, 4, 5$.

(b) The probabilities are $\frac{4}{48}$, $\frac{16}{48}$, $\frac{15}{48}$, $\frac{8}{48}$, $\frac{5}{48}$, respectively for $i = 1, 2, 3, 4, 5$.

8. At least one comes out Heads if and only if not both are Tails. The probability is therefore 3/4. Precisely one Heads occurs in two cases ((H, T) and (T, H)) out of the four equally likely outcomes, hence the probability is 1/2.

9. At least one of the dice shows a six if and only if not both dice show something else. Thus the answer is $1 - (5/6)^2$. None of the dice show a six with probability $(5/6)^2$.

10, solution 1: The event $E$ that the second die lands on a higher value than does the first consists of the following elements:

$$
\begin{aligned}
E = \{&(1,2),\ (1,3),\ (1,4),\ (1,5),\ (1,6),\\
&(2,3),\ (2,4),\ (2,5),\ (2,6),\\
&(3,4),\ (3,5),\ (3,6),\\
&(4,5),\ (4,6),\\
&(5,6)\}.
\end{aligned}
$$

Therefore $\#E = 15$, while the sample space has $\#S = 36$ elements, thus the probability is $15/36 = 5/12$.
10, solution 2: The event $E$ consists of ordered pairs of rolls such that the second member of the pair is higher than the first. Each of these pairs can be obtained by making a 2-combination of the numbers 1 through 6, and then ordering the two numbers selected. Hence $\#E = \binom{6}{2} = 15$, and $\mathbb{P}\{E\} = 15/36 = 5/12$.
10, solution 3: Define also $F$ as the event that the first die lands on a higher value than the second one. By symmetry we have $\mathbb{P}\{E\} = \mathbb{P}\{F\}$. Moreover, $(E \cup F)^{\mathrm{c}}$ is the event that the two dice show the same number, which has probability $1/6$. Therefore $5/6 = \mathbb{P}\{E \cup F\} = \mathbb{P}\{E\} + \mathbb{P}\{F\} - \mathbb{P}\{E \cap F\} = \mathbb{P}\{E\} + \mathbb{P}\{E\} - 0 = 2\mathbb{P}\{E\}$, from which $\mathbb{P}\{E\} = 5/12$.

11. Both land on red sides in $2 \cdot 2 = 4$ cases, on black sides in $2 \cdot 2 = 4$ cases, on yellow sides in 1 case, and on white sides also in 1 case. These are a total of 10 cases out of the 36 equally likely outcomes, which means a probability of $10/36 = 5/18$.

12. There are 8 equally likely possibilities for the sexes of the three children. Out of these, two outcomes mean the same sex for each of them, hence the answer is $2/8 = 1/4$.

13. The probability of at least one Heads in $n$ flips is $1 - (1/2)^n$. Thus we need to solve

$$\begin{aligned} 1 - \left(\frac{1}{2}\right)^n &\geq 0.9 \\ 0.1 &\geq \left(\frac{1}{2}\right)^n \\ \log_2(0.1) &\geq -n \\ \log_2(10) &\leq n. \end{aligned}$$

As $n$ must be an integer, our answer is that we need at least $\lceil \log_2(10) \rceil = 4$ coin flips. ($\lceil x \rceil$ here stands for the upper integer part, the smallest integer that is greater than or equal to $x$.)

14. The number of cases in which all six integers $\{1 \dots 6\}$ show up is equal to the number of permutations of these integers, which is $6!$. The sample space consists of sequences of length 6, made of any of the six integers: $\#S = 6^6$. The probability is hence $6!/6^6 = 5!/6^5 \simeq 0.015$.

15, solution with order:

(a) There are 3 ways to select the two positions, out of five, where $A$ and $B$ come so that there is one empty space between them. Once this is done, there are 2 ways to make an order between $A$ and $B$ on the two selected positions. Then there are three empty spaces left, on which we have $3! = 6$ ways to arrange $C$, $D$, and $E$. Hence there are $3 \cdot 2 \cdot 3! = 36$ arrangements with exactly one person between $A$ and $B$. Since all $5! = 120$ orders are equally likely, the answer is $36/120 = 3/10$.

(b) There are 2 ways to select the positions of $A$ and $B$, 2 ways for ordering $A$ and $B$, and $3! = 6$ ways for ordering $C$, $D$, and $E$. Hence there are $2 \cdot 2 \cdot 3! = 24$ arrangements with exactly two persons between $A$ and $B$, and the answer is $24/120 = 1/5$.

(c) There is only 1 ways to select the positions of $A$ and $B$ (they have to stand at the two ends of the line), 2 ways for ordering $A$ and $B$, and $3! = 6$ ways for ordering $C$, $D$, and $E$. Hence there are $1 \cdot 2 \cdot 3! = 12$ arrangements with exactly three persons between $A$ and $B$, and the answer is $12/120 = 1/10$.

15, solution with no order:

(a) There are 3 ways to select the two positions, out of five, where $A$ and $B$ come so that there is one empty space between them. Since all of the $\binom{5}{2}$ combinations for the two positions of $A$ and $B$ are equally likely, the probability is $3/\binom{5}{2} = 3/10$.

(b) There are 2 ways to select the two positions out of five, where $A$ and $B$ come so that there are two empty spaces between them, the probability now is $2/\binom{5}{2} = 1/5$.

(c) There is only 1 way to select the two positions out of five, where $A$ and $B$ come so that there are three empty spaces between them, the probability now is $1/\binom{5}{2} = 1/10$.

16. The number of ways 30 different birthdays can be distributed to the students is $365 \cdot 364 \cdots 336 = 365!/(365-30)!$. The number of ways any 30 birthdays can be assigned to the students is $365^{30}$. Therefore the probability is $365!/[335! \cdot 365^{30}] \simeq 0.29$.

17. The number of ways we can hit precisely two numbers is the number of ways we can select two out the five numbers drawn, times the number of ways we can select three out of the 85 numbers that are not drawn. That is, $\binom{5}{2} \cdot \binom{85}{3}$. There are $\binom{90}{5}$ different lottery tickets total, hence the probability is $\binom{5}{2} \cdot \binom{85}{3} / \binom{90}{5} \simeq 0.022$.

18. In order to capture 2 tagged deers and 2 non tagged deers, we need to select 2 out of the 5 tagged deers, and 2 out of the 15 non tagged deers. The number of ways we can do that is $\binom{5}{2} \cdot \binom{15}{2}$, while there are $\binom{20}{4}$ ways to pick any 4 deers. Therefore the probability is $\binom{5}{2} \cdot \binom{15}{2} / \binom{20}{4} \simeq 0.22$.

Alternatively, we can run the time backwards, and ask what is the probability that precisely two of our 4 deers and three out of the other 16 deers were previously tagged. In this case the answer is $\binom{4}{2} \cdot \binom{16}{3} / \binom{20}{5}$. Expanding the binomial coefficients shows that the two answers are indeed the same.

19. The probability that no psychologists are chosen is the number of ways we can choose 3 psychiatrists out of 30 divided by the number of ways to choose any 3 panelists out of 54: $\binom{30}{3} / \binom{54}{3}$. Our answer is the probability of the complement event, $1 - \binom{30}{3} / \binom{54}{3} \simeq 0.84$.