

Problems and results for the eighth week
Mathematics A3 for Civil Engineering students

1. We roll a fair die. What is the probability that we roll a 6, if we know that:
 - the roll is an even number?
 - the roll is at least 3?
 - the roll is at most 5?
2. We roll two dice. What is the probability that at least one of the dice shows a 2, if we already know that the sum of the rolls is 6? And if we do not know anything?
3. Two fair dice are rolled. What is the conditional probability that at least one lands on 6 given that the dice land on different numbers?
4. We repeatedly roll two dice at the same time, and only stop when at least one of them shows a six. What is the probability that the other also shows a six? (If you think it's $\frac{1}{6}$ then try again.)
5. There are 230 students and 30 teachers in a school. Once a flu epidemic occurs in the school, and the doctor finds the following data:

	Sick	Healthy
Boy	50	60
Girl	40	80
Teacher	10	20

A card is made for each of the 260 persons in the Doctor's office.

- (a) We randomly pick one of these cards. What is the probability that it belongs to:
 - i. a boy?
 - ii. a sick person?
 - iii. a sick boy?
 - (b) If cards of boys, girls and teachers are stored in separate drawers, and I pick one card out of the girls' drawers, what is the probability I pick a card of a sick girl?
 - (c) The Doctor's agile nurse collected all cards of sick persons into one large pile. Picking one of these randomly, what is the probability that the card belongs to a teacher?
 - (d) If I draw two cards, one after the other, out of this pile, what is the probability that the first card belongs to a boy and the second one belongs to a girl? And that both belong to boys?
6. (a) I come from a family of two children. What is the probability that the other child is my sister?
 - (b) The king comes from a family of two children. What is the probability that the other child is his sister?
 7. Each of two balls is painted either black or gold and then placed in an urn. Suppose that each ball is colored black with probability $\frac{1}{2}$, and that these events are independent.

- (a) Suppose that you obtain information that the gold paint has been used (and then at least one of the balls is painted gold). Compute the conditional probability that both balls are painted gold.
- (b) Suppose, now, that the urn tips over and one ball falls out. It is painted gold. What is the probability that both balls are gold in this case?
8. In a certain community, 36 percent of the families own a dog, and 22 percent of the families that own a dog also own a cat. In addition, 30 percent of the families own a cat.
- (a) What percentage of families own both a dog and a cat?
- (b) Out of those families who own a cat, what percentage owns also a dog?
9. An urn contains 3 red, 5 white and 6 green balls. We draw 3 balls one after the other. What is the probability that we draw a red first, a white second, and a green ball the third time, if drawings are made
- (a) with replacement,
- (b) without replacement?
10. Cockroach extinction is made in three steps in a building estate. The first step exterminates 60% of the cockroaches, but then they become more and more immune, hence the second step only exterminates 40%, and the third step 20% of them. What is the probability that a given cockroach
- (a) survives the whole procedure?
- (b) dies in the third step?
- (c) survives the whole procedure, if he was seen alive after the first step?
11. Based on our information 60% of businesses with company A , and 70% of businesses with company B turn out to be favorable. We do two businesses with the company that contacts us sooner. Each A and B has probability $1/2$ to be the first to contact us. What is the probability that
- (a) the first business we do will be favorable?
- (b) both businesses we do will be favorable?
- (c) we will do both favorable and unfavorable businesses?
12. Suppose that an ordinary deck of 52 cards (which contains 4 aces) is randomly divided into 4 hands of 13 cards each. We are interested in determining the probability that each hand has an ace. Let E_i be the event that the i th hand has exactly one ace. Determine $\mathbb{P}\{E_1E_2E_3E_4\}$ by using the multiplication rule.
13. A recent college graduate is planning to take the first three examinations in the coming summer. She will take the first exam in June. If she passes that exam, then she will take the second exam in July, and if she also passes that one, then she will take the third exam in September. If she fails an exam, then she is not allowed to take any others. The probability that she passes the first exam is 0.9. If she passes the first exam, then the conditional probability that she passes the second one is 0.8, and if she passes both the first and the second exams, then then the conditional probability that she passes the third exam is 0.7.
- (a) What is the probability that she passes all three exams?

- (b) Given that she did not pass all three exams, what is the conditional probability that she failed the second exam?
14. Drunkard Druce spends $\frac{2}{3}$ of the day in pubs. The village has 5 pubs, and Druce is not choosy, he can be found in any of the 5 pubs with equal chance. Once we set out for finding him. We have looked for him in 4 pubs already, but we haven't found him. What is the probability that he will sit in the fifth pub?
 15. 0.1% of drivers drive through the red lights of railway crossings. The crossing of that low-traffic road at the bottom of the hill is red in 5% of the time. I see from the hill that a car is just passing the crossing. What is the probability that the crossing is red at this moment?
 16. At the factory, machines are produced on production lines A , B , and C . 25% of the machines are produced on production line A , 35% on line B , and 40% on line C . It is known that 5% of machines produced on production line A , 4% of machines produced on line B , and 2% of machines produced on line C , respectively, are defective. We select randomly one machine manufactured by the factory, and it turns out to be defective. What is the probability that it was produced on line A , B , or C ?
 17. Once upon a time Odysseus met an intersection of three pathways. One of them lead to Athens, the other lead to Mycenae, and the third lead to Sparta, but he didn't know which route goes to which of these cities. He chose one of the routes by rolling a die, giving equal chance to each of these choices. He knew that, on average, Athenians only tell the truth in one case out of three, Mycenae citizens lie every second time, but people of Sparta are always honest. In the city he arrived, he asked the first man he met how much two times two was, and had four as answer. What is the probability that Odysseus finally reached Athens?
 18. Suppose that an insurance company classifies people into one of three classes: good risks, average risks, bad risks. Their records indicate that the probabilities that good, average, and bad risk persons will be involved in an accident over a 1-year span are, respectively, 0.05, 0.15, and 0.3. If 20% of the population are good risks, 50% are average risks, and 30% are bad risks, what proportion of people have accidents next year? If a policyholder had no accidents last year, what is the probability that (s)he is a good, average, or bad risk?
 19. "0" is sent with probability $\frac{1}{3}$, and "1" is sent with probability $\frac{2}{3}$ on a binary channel. Due to network noises, a sent "0" arrives as "1" with probability $\frac{1}{4}$, and a sent "1" arrives as "0" with probability $\frac{1}{5}$.
 - (a) What is the probability that a "0" arrives?
 - (b) We received a "0". What is the probability that it was also sent as a "0"?
 20. A transport company wants to determine the average number of passengers on its buses. They have to decide between two methods:
 - (a) They entrust n randomly selected passengers to count the total number of passengers on the buses they travel with. Then the company computes the average of the collected n answers.
 - (b) The company asks n of its bus drivers to count the number of passengers on their buses, and computes the average of these n answers.

Which method would you recommend? Which method will give a larger result?

Answers

1.

- $\mathbb{P}\{\text{shows a 6 and an even no.}\}/\mathbb{P}\{\text{shows an even no.}\} = \frac{1}{6}/\frac{1}{2} = \frac{1}{3}$.
- $\mathbb{P}\{\text{shows a 6 and at least 3}\}/\mathbb{P}\{\text{shows at least 3}\} = \frac{1}{6}/\frac{2}{3} = \frac{1}{4}$.
- 0.

2. $2/36$ is the probability that at least one of the dice shows a 2 and the sum is 6, and $5/36$ is the probability that the sum is 6, hence the answer is $2/5$. If no information is given, then none of the dice show a 2 with probability $\frac{5 \cdot 5}{36}$, that is, at least one of the dice is a 2 with probability $1 - \frac{5 \cdot 5}{36} = \frac{11}{36}$.

3. At least one of the numbers is 6, and the two dice show different numbers in 10 cases. The two dice show different numbers in $36 - 6 = 30$ cases. Hence the answer is $10/30 = 1/3$.

4. The question is finding the probability that both dice show a 6, given that at least one of them shows a 6. Both events occur if and only if both dice are 6, the probability of which is $1/36$. At least one shows a 6 with probability $11/36$, and so the answer is $\frac{1/36}{11/36} = \frac{1}{11}$.

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(a) i.: $[50 + 60]/260 = 11/26$. ii.: $[50 + 40 + 10]/260 = 5/13$. iii.: $50/260 = 5/26$.

(b) $40/[40 + 80] = 1/3$.

(c) $10/[50 + 40 + 10] = 1/10$.

(d) We assume that the cards are selected without replacement. The first card belongs to a boy with probability $50/[50 + 40 + 10] = 1/2$. Then the second card is of a girl with probability $40/[49 + 40 + 10] = 40/99$. The answer is the product of these two numbers, $20/99$. The probability that both cards belong to boys is $\frac{1}{2} \cdot \frac{49}{99} = 49/198$.

6.(a) Children of my family can be (in order): (b, b) , (b, g) , (g, b) , (g, g) . I'm one of the b 's with equal chance, that's four cases. Out of these four, I have a sister in two cases, and so the answer is $2/4 = 1/2$.

(b) In the (b, b) case the king can only be the older son, and in the other cases the king is one of the b 's. These are three choices of equal chance, and in two cases the other child is the king's sister. Hence the answer is $2/3$.

7.(a) The balls can be, in order, (g, g) , (g, b) , (b, g) , (b, b) , all four with equal chance. Our condition is to have at least one golden ball, which selects the first three cases. Out of these three, two golden balls occur in one case, and so the answer is $1/3$.

(b) Our condition now is that the first ball is painted gold. This selects the first two cases, one out of which is the case of two golden balls, the answer is now $1/2$.

Remark: The event that the first ball is painted gold implies that at least one of the balls is painted gold. However, these events are not the same, and so the conditional probabilities are not the same:

$$\mathbb{P}\{\cdot \mid \text{at least one of the balls is painted gold}\} \neq \mathbb{P}\{\cdot \mid \text{the first ball is painted gold}\}.$$

8. Let D be the event that a randomly selected family has a dog, and C the event that it has a cat. Given are: $\mathbb{P}\{D\} = 0.36$, $\mathbb{P}\{C\} = 0.3$, $\mathbb{P}\{C \mid D\} = 0.22$. The answers are:

(a) $\mathbb{P}\{D \cap C\} = \mathbb{P}\{C | D\} \cdot \mathbb{P}\{D\} = 0.22 \cdot 0.36 = 0.0792$, or 7.92%;

(b) $\mathbb{P}\{D | C\} = \mathbb{P}\{D \cap C\} / \mathbb{P}\{C\} = \mathbb{P}\{C | D\} \cdot \mathbb{P}\{D\} / \mathbb{P}\{C\} = 0.22 \cdot 0.36 / 0.3 = 0.264$, or 26.4%.

9. There is a total of 14 balls in the urn, hence:

(a) $\frac{3}{14} \cdot \frac{5}{14} \cdot \frac{6}{14}$,

(b) $\frac{3}{14} \cdot \frac{5}{13} \cdot \frac{6}{12}$.

10. Let E_i be the event that our hero cockroach survived the i th step, $i = 1, 2, 3$.

(a) $\mathbb{P}\{E_1 \cap E_2 \cap E_3\} = \mathbb{P}\{E_3 | E_1 \cap E_2\} \cdot \mathbb{P}\{E_2 | E_1\} \cdot \mathbb{P}\{E_1\} = 0.8 \cdot 0.6 \cdot 0.4 = 0.192$.

(b) $\mathbb{P}\{E_1 \cap E_2 \cap E_3^c\} = \mathbb{P}\{E_3^c | E_1 \cap E_2\} \cdot \mathbb{P}\{E_2 | E_1\} \cdot \mathbb{P}\{E_1\} = 0.2 \cdot 0.6 \cdot 0.4 = 0.048$.

(c) $\mathbb{P}\{E_1 \cap E_2 \cap E_3 | E_1\} = \mathbb{P}\{E_1 \cap E_2 \cap E_3\} / \mathbb{P}\{E_1\} = \mathbb{P}\{E_3 | E_1 \cap E_2\} \cdot \mathbb{P}\{E_2 | E_1\} = 0.8 \cdot 0.6 = 0.48$.

11. Let E_A be the event that company A came first, E_B the event that company B came first, F_1 and F_2 , respectively, that the first and the second business is favorable. Then

(a) $\mathbb{P}\{F_1\} = \mathbb{P}\{F_1 | A\} \cdot \mathbb{P}\{A\} + \mathbb{P}\{F_1 | B\} \cdot \mathbb{P}\{B\} = 0.6 \cdot 0.5 + 0.7 \cdot 0.5 = 0.65$.

(b) $\mathbb{P}\{F_1 \cap F_2\} = \mathbb{P}\{F_1 \cap F_2 | A\} \cdot \mathbb{P}\{A\} + \mathbb{P}\{F_1 \cap F_2 | B\} \cdot \mathbb{P}\{B\} = 0.6^2 \cdot 0.5 + 0.7^2 \cdot 0.5 = 0.425$.

(c) $\mathbb{P}\{F_1 \cap F_2^c\} + \mathbb{P}\{F_1^c \cap F_2\} = 2\mathbb{P}\{F_1 \cap F_2^c\} = 2\mathbb{P}\{F_1 \cap F_2^c | A\} \cdot \mathbb{P}\{A\} + 2\mathbb{P}\{F_1 \cap F_2^c | B\} \cdot \mathbb{P}\{B\} = 2 \cdot 0.6 \cdot 0.4 \cdot 0.5 + 2 \cdot 0.7 \cdot 0.3 \cdot 0.5 = 0.45$.

12.

$$\begin{aligned} \mathbb{P}\{E_1 E_2 E_3 E_4\} &= \mathbb{P}\{E_4 | E_1 E_2 E_3\} \cdot \mathbb{P}\{E_3 | E_1 E_2\} \cdot \mathbb{P}\{E_2 | E_1\} \cdot \mathbb{P}\{E_1\} \\ &= \frac{\binom{1}{1} \cdot \binom{12}{12}}{\binom{13}{13}} \cdot \frac{\binom{2}{1} \cdot \binom{24}{12}}{\binom{26}{13}} \cdot \frac{\binom{3}{1} \cdot \binom{36}{12}}{\binom{39}{13}} \cdot \frac{\binom{4}{1} \cdot \binom{48}{12}}{\binom{52}{13}} = \frac{4! \cdot 13 \cdot 13 \cdot 13 \cdot 13}{52 \cdot 51 \cdot 50 \cdot 49} \simeq 0.105. \end{aligned}$$

In the numerator of the above fractions we always count the number of ways how precisely one ace can be dealt to the next player. After simplifications the fraction on the right hand-side has an immediate meaning if we think about the number of ways 4 aces can be distributed into blocks of 13 each, and the number of ways 4 aces can be dealt to 52 positions without any restrictions.

13. Let A_1, A_2, A_3 the events that the student passes the first the second, and the third exam, respectively.

(a) $\mathbb{P}\{A_1 A_2 A_3\} = \mathbb{P}\{A_3 | A_1 A_2\} \cdot \mathbb{P}\{A_2 | A_1\} \cdot \mathbb{P}\{A_1\} = 0.7 \cdot 0.8 \cdot 0.9 = 0.504$.

(b) As failing the second exam ($A_1 \cap A_2^c$) is included in the event that one of the exams is unsuccessful ($\{A_1 A_2 A_3\}^c$),

$$\mathbb{P}\{A_1 \cap A_2^c | \{A_1 A_2 A_3\}^c\} = \frac{\mathbb{P}\{A_1 \cap A_2^c\}}{\mathbb{P}\{\{A_1 A_2 A_3\}^c\}} = \frac{\mathbb{P}\{A_2^c | A_1\} \cdot \mathbb{P}\{A_1\}}{1 - \mathbb{P}\{A_1 A_2 A_3\}} = \frac{(1 - 0.8) \cdot 0.9}{1 - 0.504} \simeq 0.363.$$

14. Let P_i be the event that Druce is sitting in the i^{th} pub. Then the P_i 's are mutually exclusive events, the union of which is of probability $2/3$. By Druce being non-choosy, $\mathbb{P}\{P_i\} = \frac{1}{5} \cdot \frac{2}{3} = \frac{2}{15}$. As P_5 is part of the event $P_1^c \cap P_2^c \cap P_3^c \cap P_4^c$, the probability in question is

$$\begin{aligned} \mathbb{P}\{P_5 | P_1^c \cap P_2^c \cap P_3^c \cap P_4^c\} &= \frac{\mathbb{P}\{P_5\}}{\mathbb{P}\{P_1^c \cap P_2^c \cap P_3^c \cap P_4^c\}} = \frac{\mathbb{P}\{P_5\}}{\mathbb{P}\{\{P_1 \cup P_2 \cup P_3 \cup P_4\}^c\}} \\ &= \frac{\mathbb{P}\{P_5\}}{1 - \mathbb{P}\{P_1\} - \mathbb{P}\{P_2\} - \mathbb{P}\{P_3\} - \mathbb{P}\{P_4\}} \\ &= \frac{2/15}{1 - 2/15 - 2/15 - 2/15 - 2/15} = \frac{2}{7}. \end{aligned}$$

15. Let D be the event that the arriving driver drives through the railway crossing, R the event that the crossing shows a red light at that moment. Given are $\mathbb{P}\{R\} = 0.05$, $\mathbb{P}\{D | R\} = 0.001$. We can also assume that the driver surely drives through if the lights are not red: $\mathbb{P}\{D | R^c\} = 1$. By Bayes' Theorem

$$\mathbb{P}\{R | D\} = \frac{\mathbb{P}\{D | R\} \cdot \mathbb{P}\{R\}}{\mathbb{P}\{D | R\} \cdot \mathbb{P}\{R\} + \mathbb{P}\{D | R^c\} \cdot \mathbb{P}\{R^c\}} = \frac{0.001 \cdot 0.05}{0.001 \cdot 0.05 + 1 \cdot 0.95} \simeq 0.00005,$$

which is practically zero. There is a negligible chance that the driver passes on red, compared to the chance of him passing lawfully.

18. Let G, A, B be the event that a randomly selected person is good, average or bad risk. These form a partition of the sample space. Let also C be the event that the random person gets involved in an accident in a given year. Then

$$\mathbb{P}\{C\} = \mathbb{P}\{C | G\} \cdot \mathbb{P}\{G\} + \mathbb{P}\{C | A\} \cdot \mathbb{P}\{A\} + \mathbb{P}\{C | B\} \cdot \mathbb{P}\{B\} = 0.05 \cdot 0.2 + 0.15 \cdot 0.5 + 0.3 \cdot 0.3 = 0.175.$$

The answers for the second question are:

$$\mathbb{P}\{G | C^c\} = \mathbb{P}\{C^c | G\} \cdot \frac{\mathbb{P}\{G\}}{\mathbb{P}\{C^c\}} = [1 - \mathbb{P}\{C | G\}] \cdot \frac{\mathbb{P}\{G\}}{1 - \mathbb{P}\{C\}} = [1 - 0.05] \cdot \frac{0.2}{1 - 0.175} \simeq 0.23.$$

Similarly, $\mathbb{P}\{A | C^c\} \simeq 0.52$, $\mathbb{P}\{B | C^c\} \simeq 0.25$.

19. Let S_i be the event that i is sent on the channel ($i = 0, 1$), R_i the event that i is received. Given are $\mathbb{P}\{S_0\} = 1/3$, $\mathbb{P}\{S_1\} = 2/3$, $\mathbb{P}\{R_1 | S_0\} = 1/4$, $\mathbb{P}\{R_0 | S_1\} = 1/5$.

(a)

$$\begin{aligned} \mathbb{P}\{R_0\} &= \mathbb{P}\{R_0 | S_0\} \cdot \mathbb{P}\{S_0\} + \mathbb{P}\{R_0 | S_1\} \cdot \mathbb{P}\{S_1\} \\ &= [1 - \mathbb{P}\{R_1 | S_0\}] \cdot \mathbb{P}\{S_0\} + \mathbb{P}\{R_0 | S_1\} \cdot \mathbb{P}\{S_1\} = \left[1 - \frac{1}{4}\right] \cdot \frac{1}{3} + \frac{1}{5} \cdot \frac{2}{3} = \frac{23}{60}. \end{aligned}$$

(b)

$$\mathbb{P}\{S_0 | R_0\} = \mathbb{P}\{R_0 | S_0\} \cdot \frac{\mathbb{P}\{S_0\}}{\mathbb{P}\{R_0\}} = \left[1 - \frac{1}{4}\right] \cdot \frac{1/3}{23/60} = \frac{15}{23}.$$

20. Different quantities can be measured by the two methods, the average number of passengers on a bus is given by method (b). Method (a) is suitable for measuring the average crowd seen by a passenger. In this method we have a greater chance to sample a more crowded bus, while method (b) gives equal chance to each bus. Hence method (a) can be expected to give a larger result than method (b). (That is: as a typical passenger we see more crowd than as a typical bus driver.)