

## PROBABILITY, Lesson 1. Combinatorial Analysis

1. **Permutations.** How many different orders of  $n$  objects exist?

- Without repetition (there are  $n$  different objects):  $n!$
- With repetition (there are  $n$  objects of which  $n_1, \dots, n_r$  are alike):  $\frac{n!}{n_1! \dots n_r!}$

2. **Variations.** How many different orders of  $k$  objects chosen out of  $n$  objects exist?

- Without repetition (an object is chosen at most once):  $n(n-1) \dots (n-k+1) = \frac{n!}{(n-k)!}$
- With repetition (an object may be chosen several times):  $n^k$

3. **Combinations.** How many ways  $k$  objects can be chosen out of  $n$  objects?

- Without repetition (an object is chosen at most once):  $\frac{n!}{k!(n-k)!} = \binom{n}{k}$ ,  $k \leq n$
- With repetition (an object may be chosen several times):  $\binom{k+n-1}{k}$ ,  $k \in \mathbb{N}$

Identities containing binomial coefficients:

$$(a+b)^n = \sum_{k=0}^n \binom{n}{k} a^k b^{n-k} \qquad 2^n = \sum_{k=0}^n \binom{n}{k} \qquad 0 = \sum_{k=0}^n (-1)^k \binom{n}{k}$$

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k} \qquad \binom{n+m}{k} = \sum_{i=0}^k \binom{n}{i} \binom{m}{k-i}$$

### Probability Space

$(\Omega, \mathcal{A}, \mathbb{P})$ , where  $\Omega$  is the *sample space* (set of all possible outcomes=*elementary events*),  $\mathcal{A} = \{A \mid A \subset \Omega\}$  is the Boole-algebra of the all possible events (including  $\emptyset$ =impossible/null event and  $\Omega$ =certain/sure event), and the set function  $\mathbb{P} : \mathcal{A} \rightarrow \mathbb{R}$  satisfies the following **AXIOMS**:

1. For any  $A \in \mathcal{A}$ :  $0 \leq \mathbb{P}(A) \leq 1$
2.  $\mathbb{P}(\Omega) = 1$
3. For any sequence of mutually exclusive events  $A_1, A_2, \dots$ :  $\mathbb{P}(\sum_i A_i) = \sum_i \mathbb{P}(A_i)$

Propositions implied by the axioms:

- $\mathbb{P}(\bar{A}) = 1 - \mathbb{P}(A)$
- Probability is a monotonous set function: if  $A \subset B$ , then  $\mathbb{P}(A) \leq \mathbb{P}(B)$ .
- $\mathbb{P}(\sum_{i=1}^n A_i) = \sum_{k=1}^n (-1)^{k-1} S_k$ ,  $S_k = \sum_{1 \leq i_1 < i_2 < \dots < i_k \leq n} \mathbb{P}(A_{i_1} A_{i_2} \dots A_{i_k})$  (inclusion-exclusion)

Examples for probability spaces:

- *Combinatorial*: the sample space has finite number of equally like outcomes,  $\mathbb{P}(A) = |A|/|\Omega|$ .
- *Geometrical*: the sample space is a region with finite measure  $\mu$  (length, area, volume),  $\mathbb{P}(A) = \mu(A)/\mu(\Omega)$ .