

POBABILITY, Lesson 2.
Conditional Probability, Independence

- *Definition.* $\mathbb{P}(A|B) = \frac{\mathbb{P}(AB)}{\mathbb{P}(B)}$, $\mathbb{P}(B) > 0$.

With B fixed, $\mathbb{Q}(A) := \mathbb{P}(A|B)$. $(\Omega, \mathcal{A}, \mathbb{Q})$ is also a probability space with all of its consequences.

- *Definition.* B_1, B_2, \dots is a complete set of mutually exclusive events, if $B_i B_j = \emptyset$ ($i \neq j$) and $\sum_i \mathbb{P}(B_i) = 1$.
- **Theorem.** Let B_1, B_2, \dots be a complete set of mutually exclusive events and A be an arbitrary event. Then

$$\mathbb{P}(A) = \sum_i \mathbb{P}(A|B_i) \cdot \mathbb{P}(B_i).$$

- **Theorem (Bayes).** Let B_1, B_2, \dots be a complete set of mutually exclusive events and A be an arbitrary event. Then

$$\mathbb{P}(B_k|A) = \frac{\mathbb{P}(A|B_k) \cdot \mathbb{P}(B_k)}{\sum_i \mathbb{P}(A|B_i) \cdot \mathbb{P}(B_i)}, \quad k = 1, 2, \dots$$

- **Theorem (factorization).** Let A_1, A_2, \dots, A_n be arbitrary events. Then

$$\mathbb{P}(A_1 A_2 \dots A_n) = \mathbb{P}(A_1) \cdot \mathbb{P}(A_2|A_1) \dots \mathbb{P}(A_n|A_1 \dots A_{n-1}).$$

- *Definition.* A and B are independent, if

$$\mathbb{P}(AB) = \mathbb{P}(A) \cdot \mathbb{P}(B).$$

Remark: if $\mathbb{P}(A) \neq 0$ and $\mathbb{P}(B) \neq 0$, then A and B cannot be exclusive and independent at the same time.

- *Definition.* The events A_1, \dots, A_n are (completely) independent, if

$$\mathbb{P}(A_{i_1} \dots A_{i_k}) = \mathbb{P}(A_{i_1}) \dots \mathbb{P}(A_{i_k})$$

for any k -tuple A_{i_1}, \dots, A_{i_k} and $k = 2, \dots, n$. ($k = 2$ case: pairwise independence, weaker than independence.)

- **Theorem (Borel–Cantelli Lemma) FACULTATIVE**

1. Let A_1, A_2, \dots be arbitrary events such that $\sum_{n=1}^{\infty} \mathbb{P}(A_n) < \infty$. Then

$$\mathbb{P}(\limsup_{n \rightarrow \infty} A_n) = 0.$$

2. Let A_1, A_2, \dots be (completely) independent events such that $\sum_{n=1}^{\infty} \mathbb{P}(A_n) = \infty$. Then

$$\mathbb{P}(\limsup_{n \rightarrow \infty} A_n) = 1.$$

- **Theorem (Lovász Local Lemma).** Let A_1, A_2, \dots be events with dependency graph $G = (V, E)$, where $V = \{v_1, \dots, v_n\}$ corresponds to the events and A_i is independent of $\{A_j \mid \{v_i, v_j\} \notin E\}$. Suppose that

$$\mathbb{P}(A_i) \leq \frac{1}{4d_{\max}(G)} \quad (i = 1, \dots, n),$$

where $d_{\max}(G)$ is the maximum vertex degree in G . Then

$$\mathbb{P}(\overline{A_1} \dots \overline{A_n}) > 0.$$