Math. A4, Midterm Test 2 Sample, 2024

1. We have 100 electric bulbs, the lifetime of which follows i.i.d. exponential distribution with expectation 5 hours. They are used one after the other continuously. Estimate the probability that after 525 hours we still have bulbs that were not used until that time.

2. Consider the following joint density of X and Y:

$$f(x,y) = \begin{cases} 24xy & \text{if } 0 < x,\ 0 < y,\ x+y < 1 \\ 0 & \text{otherwise.} \end{cases}$$

   (a) Calculate the marginal distributions! Are X and Y independent?

   (b) Find the $Y > X$ probability!

   (c) Find the p.d.f. of the conditional distribution of Y conditioned on $X = x$! Find the $\mathbb{E}(Y|X = x)$ conditional expectation!

   (d) Find $\mathrm{Cov}(X, Y)$!

3. Calculate the p.d.f. of the following random variable: If X is exponentially distributed with parameter λ, then find the p.d.f. of $Y := 4X - 5$.

4. We have the following discrete distribution: $\mathbb{P}_\theta(X = 0) = \frac{5\theta}{4}$, $\mathbb{P}_\theta(X = 2) = \frac{2-2\theta}{4}$, $\mathbb{P}_\theta(X = 4) = \frac{1-2\theta}{4}$, $\mathbb{P}_\theta(X = 5) = \frac{1-\theta}{4}$. Give a Maximum Likelihood estimate for θ from a sample of $0, 2, 4, 4, 5, 2, 5, 5$.

5. Let $X_1, \ldots, X_n$ be i.i.d. sample from the distribution with p.d.f. $f_\theta(x) = \frac{1}{\theta} x^{\frac{1-\theta}{\theta}}$ (if $0 < x < 1$, otherwise 0). Find the ML-estimate of $\theta > 0$.

6. Let (X, Y) have bivariate normal distribution with expectation vector $\boldsymbol{\mu} = (-3, 2)$ and covariance matrix

$$\boldsymbol{C} = \begin{pmatrix} 4 & 1 \\ 1 & 5 \end{pmatrix}.$$

   Find the distribution of $2X - Y$.