PROBABILITY A4, Problems to Lesson 2.

1. *Matching problem*: Suppose that each of n students brings a present to a party. The presents are mixed up and then each student randomly selects a present. What is the probability that none of the students selects his/her own present? (Equivalent problem: n married couples are dancing: n man–woman pairs are formed randomly. What is the probability that none of the men selects his own wife?)

2. How many fair coins should be tossed so that the probability of having at least one head be more than 0.9?

3. *An infinite sequence of independent trials* is performed (e.g., having a 5-hit with LOTTO; there is a fire event in a certain college within a year). Each trial results in a *success* with probability p and a *failure* with probability $1 - p$. What is the probability that at least 1 success occurs in the first n trials; exactly k successes occur in the first n trial? What is the probability that in the infinite sequence there are infinitely many successes (there are infinitely many fire events); all trials result in failure (no fire at all)?

4. By my knowledge, the neighboring family with two children has at least one boy. On this condition, what is the probability that both children are boys?

5. There are 3 pubs in our little town and I am looking for my best friend, who is in pub with probability 60%. I have not found him in the first pub, and I have not found him in the second one. On this condition, what is the probability that I shall find him in the third pub?

6. Mosquitoes are sprayed in three steps. In the first step, they get rid of 70%, in the second step they get rid of 50%, and in the third step they get rid of 30% of the existing mosquitoes. What is the probability that a mosquito survives all the three sprayings?

7. 0.1% of the population suffers of a certain illness. A medical test for it works with error: sometimes it diagnoses a healthy person ill, or an ill person healthy; both kinds of error happen with probability 0.01. Having a positive test result (indicating illness), what is the probability that I am still healthy? Having two positive results (of two independent medical tests), what is the probability that I am still healthy?

8. **Secretary Problem** (Marriage or Sultan's dowry problem). Secretaries are interviewed one after the other in the following way:

   - There is one secretarial position available.
   - The number n of applicants is known.
   - The applicants are interviewed sequentially in random order, each order being equally likely.
   - It is assumed that you can rank all the applicants from best to worst without ties. The decision to accept or reject an applicant must be based only on the relative ranks of those applicants interviewed so far.
   - An applicant once rejected cannot later be recalled.

   What is the probability that you find the best one by the following (m-)strategy: for some integer $0 < m < n$ you reject the first m applicants, and then choose the next applicant who is best in the relative ranking of the observed applicants. What is the optimal m? (It can be shown that this is the best strategy.)

9. (**BONUS**) There are 3 prisoners in the prison. Two of them will be executed tomorrow. The custodian knows who won't be executed. One prisoner asks the custodian to tell a name of the two others who will be surely executed. The custodian tells it. The prisoner thinks that the original probability that he won't be executed is $\frac{1}{3}$, but after this information it becomes $\frac{1}{2}$ as either he will be executed or the third one (whose name is not mentioned by the custodian). Does this information indeed increase his chances?

10. (**BONUS**) In a game there are 3 doors, behind one of them there is a car, otherwise a goat. If you point randomly to a closed door, you win the car with probability $\frac{1}{3}$. The organizer knows what is behind a door and changes the game as follows: if you point to a closed door, he opens you another one with a goat behind it (it is always possible). After this, he offers you that you can change your original choice. Is it worth for you to change your mind? Whether it will increase your chances to get the car?

11. In Budapest, 36% of the families has a dog and 30% has a cat. Among the families, having a dog, 22% has a cat too.

    (a) What proportion of the families has both dog and cat?

    (b) Among the families, having a cat, what proportion has a dog too?

12. Let A and B be two independent events, $\mathbb{P}(A) = 0.3$ and $\mathbb{P}(B) = 0.8$. What is the probability that

    (a) both A and B happen;

    (b) exactly one of them happens;

    (c) A happens, if B has already happened.

    (d) A happens, provided B has not happened.

13. At a conference there are 30 mathematicians and 24 physicists. Out of the participants 3 are chosen randomly to take part at a discussion. What is the probability that there will be at least one physicist among them?

14. Out of 5 women and 9 men a committee is to be form that contains 3 women and 4 men.

    (a) How many different committees can be formed?

    (b) How many different committees can be formed if there are two men who cannot work together in a committee?

15. 4 towns are connected by roads as follows: Both A and D are connected to B and C (A and D are not connected). B and C are also connected. In a winter day the roads, independently of each-other, are blocked by snow with probability $\frac{1}{5}$. What is the probability that someone can get from A to D in some way that day?

16. Three cooks, A, B, and C bake a special kind of cake, and with respective probabilities 0.03, 0.05, and 0.06 it fails to rise. In the restaurant where they work, A bakes 50 per cent of these cakes, B 30 per cent, and C 20 per cent. What is the average proportion of failures? What proportion of „failures" is caused by A? by B? by C?