PROBABILITY AND STATISTICS, Problems to Lesson 5.

1. Which of the following functions can be c.d.f.'s of a continuous r.v.?

   (a) $$F(x) = \begin{cases} 2 - \dfrac{2}{x+1} & \text{, if } x \geq 0, \\ 0 & \text{, otherwise} \end{cases}$$

   (b) $$F(x) = \begin{cases} 1 - \mathrm{e}^{-x} & \text{, if } x \geq 0, \\ 0 & \text{, otherwise} \end{cases}$$

   (c) $$F(x) = \begin{cases} 0 & \text{, if } x \leq 0, \\ \dfrac{x}{4} \cdot (4 - x) & \text{, if } 0 < x \leq 2, \\ 1 & \text{, if } x > 2 \end{cases}$$

2. Which of the following functions can be p.d.f.'s of a continuous r.v.?

   (a) $$f(x) = \begin{cases} \dfrac{2}{x} & \text{, if } x > 1, \\ 0 & \text{, otherwise} \end{cases}$$

   (b) $$f(x) = \begin{cases} \dfrac{\sin(x)}{2} & \text{, if } 0 < x < 2, \\ 0 & \text{, otherwise} \end{cases}$$

   (c) $$f(x) = \begin{cases} 2\mathrm{e}^{-2x} & \text{, if } x > 0, \\ 0 & \text{, otherwise} \end{cases}$$

3. Find the expectation and variance of the r.v. X with p.d.f.

   $$f(x) = \begin{cases} 2x & \text{, if } 0 < x < 1, \\ 0 & \text{, otherwise} \end{cases}$$

   Find the probability of $X \geq \frac{1}{2}$. Find the expectation, variance, and median of X.

4. A gas station gets their shipment of oil once a week. If their weekly sales (measured in 1000 liters) is a random variable with density

   $$f(x) = \begin{cases} 5 \cdot (1 - x)^4, & 0 < x < 1 \\ 0, & \text{otherwise}, \end{cases}$$

   then how big of a shipment do they need weekly if they want a less than 0.01 probability to run out of oil in a given week?

5. The lifetime of a TV (in years) is exponential with parameter $\lambda = 1/8$. If someone buys a used one, what is the probability that he can use it for more than 8 years? For what time to give a guarantee if they want to treat only 5% of the TV's within that time?

6. The lifetime (year) of a radioactive isotope, starting decay in 1986, be X. X is exponentially distributed with parameter $\lambda = \frac{1}{140}$ ($\mathbb{E}(X) = 140$). Questions with solutions:

   (a) Find the halving time of the isotope (median of the distribution): it is T such that $F(T) = \frac{1}{2}$. From here, $T = \frac{\ln 2}{\lambda} = 140 \ln 2 < 140$.

   (b) Which proportion of the isotopes is present now?

   $$\mathbb{P}(X > 2024-1986) = \mathbb{P}(X > 38) = 1-F(38) = 1-(1-e^{\frac{1}{140}38}) = e^{-\frac{38}{140}}$$

   Which proportion of the isotopes present at 2000, is also present now?

   $$\mathbb{P}(X > 38|X > 14) = \mathbb{P}(X > 24) = 1-F(17) = 1-(1-e^{\frac{1}{140}24}) = e^{-\frac{24}{140}}$$

7. Let $X \sim \mathcal{N}(\mu, \sigma)$ be Gaussian r.v. Using *standardization* $Y = \frac{X-\mu}{\sigma}$, find the following probabilities:

$$\mathbb{P}\left(\mu - \sigma < X < \mu - \sigma\right) = \mathbb{P}\left(-\sigma < X - \mu < \sigma\right) = \mathbb{P}\left(-1 < \frac{X-\mu}{\sigma} < 1\right)$$
$$= \Phi(1) - \Phi(-1) = \Phi(1) - (1 - \Phi(1)) = 2\Phi(1) - 1 \approx 0.68$$

The value of $\Phi(1) \approx 0.84$ is obtained from the standard normal table. As well,

$$\mathbb{P}(\mu - 2\sigma < X < \mu - 2\sigma) = \mathbb{P}(-2 < \frac{X-\mu}{\sigma} < 2) = \Phi(2) - \Phi(-2) = \Phi(2) - (1 - \Phi(2))$$
$$= 2\Phi(2) - 1 \approx 0.95.$$

The value of $\Phi(2) \approx 0.9772$ is obtained from the standard normal table.

$\Phi(z)$

z	**0.00**	**0.01**	**0.02**	**0.03**	**0.04**	**0.05**	**0.06**	**0.07**	**0.08**	**0.09**
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998
3.5	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998
3.6	0.9998	0.9998	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.7	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.8	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.9	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000