PROBABILITY AND STATISTICS, Problems to Lesson 6.

1. Let X be a continuous random variable with p.d.f. $f(t) = a(2-t-t^2)$ if $-2 \le t \le 1$ and 0, otherwise.

   (a) Find the value of a and the c.d.f. of X.

   (b) Calculate $\mathbb{P}(-1 < X \le 0.5)$ and the expectation and variance of X.

2. I know from experience that if I start looking for squirrels at this time of the day, I will spot the first one in half an hour on average.

   (a) What is the probability that I don't see any for more than an hour?

   (b) Assuming that I have been waiting for an hour now, what is the chance that I will need to wait at least one more hour?

   (c) For hedgehogs, I need to wait for 40 minutes on average. Hedgehogs and squirrels are assumed to be independent. What is the chance that I see neither animal for the next hour?

   (d) What is the expected time to see my first animal (of either type)?

3. (**BONUS**) A bus goes between city A and city B, which are 100km from each other. The bus breaks down quite often, at a random location that is uniformly distributed along the route. If it breaks down, they call the closest repair center. Currently, the repair centers are in the 2 cities, and one halfway in between them. It has been suggested that it would be better to place the repair centers at 25km, 50km and 75km. Do we agree? Why? Under better placement it is understood that the distance between the repair facility and the bus is shorter on average.

4. Most IQ tests follow a Normal Distribution with a mean of 100 points and a standard deviation of 15 points. If we believe that these tests are accurate, then (a) What percentage of people have an IQ between 95 and 110? (b) How big of an interval around 100 points does 50% of the population fall into? (c) In towns with 2500 people, on average, how many people will have at least 125 IQ?

5. A random glass in the university cafeteria breaks at an exponentially distributed time, with mean 24 months. What is the probability that:

   (a) Out of 5 glasses, at most 3 breaks within a year?

   (b) Out of 500 glasses, at most 210 breaks within a year?

6. (**Bonus**) In Monte Carlo integration, we are given a continuous function $g : [0,1] \longrightarrow \mathbb{R}_+$ and we would like to estimate the value of $\int_0^1 g(x)\,dx$. If $X \sim \mathcal{U}(0,1)$, then $\int_0^1 g(x)\,dx = \mathbb{E}(g(X))$. By the law of large numbers, we approximate this with the sample mean: let $X_1, \ldots, X_n \sim \mathcal{U}(0,1)$ be i.i.d. rv's and so, $Y_i := g(X_i)$ for $i = 1, \ldots, n$. Then the approximation of the above integral is $\bar{Y} = \frac{1}{n}\sum_{i=1}^n Y_i = \sum_{i=1}^n g(X_i)$. The quality of approximation increases with n.

   How large n should be if we want to get an approximation error at most 0.1 with probability at least 0.98 if $g(x) = x^3$?

   (The problem can be generalized to definite integrals over any finite interval $[a,b]$ and to continuous functions taking on negative values.)