PROBABILITY A4, Problems to Lessons 8-9.

1. The lamp on a police car circulates with constant speed. Find the distribution of the projection of the light on the wall in 1 unit distance from the car. More precisely: let $X \sim \mathcal{U}(-\frac{\pi}{2}, \frac{\pi}{2})$ be rv. Find the distribution of the rv $Y = \tan X$. Does the expectation of Y exist?

2. *(Rendezvous problem).* Romeo and Julia decide to meet at a given location, where they arrive randomly and independently of each other between midnight and 1 am. They decide to wait for the other 20 minutes, then to leave. Find the probability, that they will meet each other under this condition.

3. X and Y cast a fair die independently. Find the distribution of the minimum and of the maximum of the two outcomes.

4. Let X and Y be i.i.d. $\mathcal{U}(0,1)$ rv's. Find the distribution of
   (a) $\min\{X, Y\}$,
   (b) $\max\{X, Y\}$,
   (c) $X + Y$,
   (d) XY,
   (e) Y/X.

5. Let $X \sim \mathcal{U}(0,1)$. Find the distribution of X^2.

6. Let $X \sim \mathcal{U}(0,1)$. Find the distribution of $\sqrt{X}$.

7. Let $X \sim \mathcal{U}(-1,1)$. Find the distribution of X^2.

8. Let $X \sim \mathcal{N}(0,1)$. Find the distribution of X^2.

9. Let $X \sim \mathcal{N}(0,1)$. Find the distribution of $aX + b$, where $b \in \mathbb{R}$ and $a > 0$ are given constants.

10. Let X and Y be i.i.d $\mathcal{N}(0,1)$ rv's. Find the distribution of $X + Y$!

11. Let $X_1, X_2, \ldots \sim \mathsf{Exp}(\lambda)$ be i.i.d. exponentially distributed random variables.
    (a) Find the distribution of $Z = X_1 + X_2$?
    (b) What is the probability that $X_1 + X_2 > t$, where $t > 0$ is given real?
    (c) What is the expectation and variance of $X_1 + X_2 + \cdots + X_n$?
    (d) Find the distribution of $\min\{X_1, X_2, \ldots, X_n\}$!
    (e) Find the distribution of $\max\{X_1, X_2, \ldots, X_n\}$!

12. Let $X \sim \mathsf{Exp}(\lambda)$ and $Y \sim \mathsf{Exp}(\nu)$ be independent, exponentially distributed random variables. Find the distribution of $Z = X + Y$?

13. Let $X \sim \mathcal{P}(\lambda)$ and $Y \sim \mathcal{P}(\mu)$ be independent rv's. Find the distribution of $X + Y$!

14. Queuing in the supermarket takes for me a uniform random time between 5 and 12 minutes. Then, in the coffee-shop, I spend an independent uniform random time between 2 and 4 minutes with queuing.

    (a) What is the density function of the total time I spend with queuing?

    (b) What is the chance I spend more than 10 minutes with queuing?

    (c) What is the expectation and variance of my queuing time?

15. Consider the following joint densities of X and Y:

    (a)
    $$f(x,y) = \begin{cases} 4xy & \text{if } 0 < x < 1, 0 < y < 1 \\ 0 & \text{otherwise.} \end{cases}$$

    (b)
    $$f(x,y) = \begin{cases} A \cdot (x^2 y + y^2 x), & 0 < x < 1, 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

    where A is a positive constant that first must be calculated.

    (c) (X, Y) is uniformly distributed on $\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$.

    (d)
    $$f(x,y) = \begin{cases} 2, & 0 < y < x < 1 \\ 0 & \text{otherwise.} \end{cases}$$

    - Calculate the marginal distributions! Are X and Y independent?
    - Find $\mathbb{P}(X > \frac{1}{2}, Y < \frac{1}{2})$ for (a).
    - Find $\mathbb{P}(X > \frac{1}{2} \,|\, Y < \frac{1}{2})$ for (b) and (d).
    - Find $\mathbb{P}(X^2 + Y^2 < \frac{1}{4})$ for (c).
    - Find the p.d.f. of the conditional distribution of Y conditioned on $X = x$! Find the $\mathbb{E}(Y|X = x)$ conditional expectation!
    - Find the distribution of $X + Y$!