PROBABILITY A4, Lesson 1.
Combinatorial Analysis

1. **Permutations.** How many different orders of n objects exist?
   - Without repetition (there are n different objects): $n!$ (Stirling formula: $(\frac{n}{e})^n\sqrt{2\pi n}$)
   - With repetition (there are n objects of which $n_1, \ldots, n_r$ are alike): $\frac{n!}{n_1!\ldots n_r!}$
2. **Variations.** How many different orders of k objects selected from a set of n (different) objects exist?
   - Without repetition (an object is selected at most once): $n(n-1)\ldots(n-k+1) = \frac{n!}{(n-k)!}$
   - With repetition (an object may be selected several times): n^k
3. **Combinations.** How many different ways k objects can be selected from a set of n (different) objects?
   - Without repetition (an object is selected at most once): $\frac{n!}{k!(n-k)!} = \binom{n}{k}, \quad k \le n$
   - With repetition (an object may be selected several times): $\binom{k+n-1}{k}, \quad k \in \mathbb{N}$

Identities containing binomial coefficients:

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} \qquad 2^n = \sum_{k=0}^{n} \binom{n}{k} \qquad 0 = \sum_{k=0}^{n} (-1)^k \binom{n}{k}$$

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k} \qquad \binom{n+m}{k} = \sum_{i=0}^{k} \binom{n}{i}\binom{m}{k-i} \quad \text{if } k \le \min\{n, m\}$$

Multinomial theorem:

$$(a_1 + \cdots + a_r)^n = \sum_{n_1,\ldots,n_r \ge 0,\, n_1+\cdots+n_r=n} \binom{n}{n_1, \ldots, n_r} a_1^{n_1} \ldots a_r^{n_r},$$

where $a_1, \ldots, a_r \in \mathbb{R}$ and the multinomial coefficient is defined by

$$\binom{n}{n_1, \ldots, n_r} := \frac{n!}{n_1! \ldots n_r!}$$

In particular, if $a_1 = \cdots = a_r = 1$, then

$$r^n = \sum_{n_1,\ldots,n_r \ge 0,\, n_1+\cdots+n_r=n} \binom{n}{n_1, \ldots, n_r},$$

where there are $\binom{n+r-1}{n}$ terms in the summand, i.e., there are $\binom{n+r-1}{n}$ distinct nonnegative integer-valued vectors $(n_1, \ldots, n_r)$ satisfying $n_1 + \cdots + n_r = n$.

The above formula gives a relation between variations and combinations with repetition.

Probability Space

$(\Omega, \mathcal{A}, \mathbb{P})$, where Ω is the *sample space* (set of all possible outcomes=*elementary events*), $\mathcal{A} = \{A \,|\, A \subset \Omega\}$ is the σ-algebra of the all possible events (including $\emptyset$=impossible/null event and Ω=certain/sure event), and the set function $\mathbb{P} : \mathcal{A} \to \mathbb{R}$ satisfies the following **AXIOMS**:

1. For any $A \in \mathcal{A}$: $\quad 0 \leq \mathbb{P}(A) \leq 1$
2. $\mathbb{P}(\Omega) = 1$
3. For any sequence of mutually exclusive events $A_1, A_2, \ldots$: $\quad \mathbb{P}(\sum_i A_i) = \sum_i \mathbb{P}(A_i)$

Propositions implied by the axioms:

- $\mathbb{P}(\overline{A}) = 1 - \mathbb{P}(A)$
- Probability is a monotonous set function: if $A \subset B$, then $\mathbb{P}(A) \leq \mathbb{P}(B)$.
- $\mathbb{P}(\sum_{i=1}^n A_i) = \sum_{k=1}^n (-1)^{k-1} S_k, \quad S_k = \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} \mathbb{P}(A_{i_1} A_{i_2} \ldots A_{i_k})$ (inclusion–exclusion)
- Probability is a continuous set function: if $\lim_{n\to\infty} A_n$ exists, then $\mathbb{P}(\lim_{n\to\infty} A_n) = \lim_{n\to\infty} \mathbb{P}(A_n)$. ($\lim_{n\to\infty} A_n$ exists, if $\limsup_{n\to\infty} A_n = \liminf_{n\to\infty} A_n$, where $\limsup_{n\to\infty} A_n = \prod_{n=1}^{\infty} \sum_{i=n}^{\infty} A_i$, $\liminf_{n\to\infty} A_n = \sum_{n=1}^{\infty} \prod_{i=n}^{\infty} A_i$.)

Examples for probability spaces:

- *Combinatorial*: the sample space has finite number of equally like outcomes, $\mathbb{P}(A) = |A|/|\Omega|$.
- *Geometrical*: the sample space is a region with finite measure μ (length, area, volume), $\mathbb{P}(A) = \mu(A)/\mu(\Omega)$.