PROBABILITY A4, Lesson 6-7.

- **Useful Inequalities**

  1. *Markov's Inequality:* Let X be a r.v. of finite first moment and taking on nonnegative values. Then
  $$\mathbb{P}(X \geq c) \leq \frac{\mathbb{E}(X)}{c}, \qquad \forall c > 0.$$
  2. *Chebyshev's Inequality:* Let X be a r.v. with finite second moment. Then
  $$\mathbb{P}(|X - \mathbb{E}(X)| \geq \varepsilon) \leq \frac{\mathrm{Var}(X)}{\varepsilon^2}, \qquad \forall \varepsilon > 0.$$

- **Laws of Large Numbers, Central Limit Theorem**

  1. *Weak Law*: If $X_1, \dots, X_n$ are i.i.d. with finite $\mathbb{E}(X_i) = \mu$ and $\mathrm{Var}(X_i) = \sigma^2$, then $\overline{X}_n \to \mu$ in probability:
  $$\lim_{n\to\infty} \mathbb{P}(|\overline{X}_n - \mu| > \varepsilon) = 0, \quad \forall \varepsilon > 0, \text{ where } \overline{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i.$$
  Indeed, by the Chebyshev's Inequality
  $$\mathbb{P}(|\overline{X}_n - \mu| > \varepsilon) \leq \frac{\sigma^2}{n\varepsilon^2}.$$
  2. *Strong Law*: If $X_1, \dots, X_n$ are i.i.d. with finite $\mathbb{E}(X_i) = \mu$, then $\overline{X}_n \to \mu$ almost surely:
  $$\mathbb{P}(\lim_{n\to\infty} \overline{X}_n = \mu) = 1.$$
  3. **CLT**: If $X_1, \dots, X_n$ are i.i.d. with finite $\mathbb{E}(X_i) = \mu$ and $\mathrm{Var}(X_i) = \sigma^2$, then
  $$\frac{\sum_{i=1}^{n} X_i - n\mu}{\sqrt{n}\sigma} \Rightarrow \mathcal{N}(0,1) \quad \text{in distribution (convergence of c.d.f.'s)}, \quad n \to \infty.$$
  Equivalently,
  $$\frac{\overline{X}_n - \mu}{\sigma}\sqrt{n} \Rightarrow \mathcal{N}(0,1) \quad \text{in distribution (convergence of c.d.f.'s)}, \quad n \to \infty.$$
  Applying it to $X_1, X_2, \dots$ i.i.d. Bernoulli with parameter p: for large n, $\sum_{i=1}^{n} X_i$ is approximately $\mathcal{N}(np, \sqrt{np(1-p)})$ (also called De Moivre–Laplace theorem) and $\overline{X}_n$ is approximately $\mathcal{N}(p, \sqrt{\frac{p(1-p)}{n}})$.
  Also, in this case, the weak low of large numbers boils down to
  $$\mathbb{P}(|\overline{X}_n - \mu| > \varepsilon) \leq \frac{p(1-p)}{n\varepsilon^2} \leq \frac{1}{4n\varepsilon^2}. \tag{1}$$