2.1 Introduction

- Introduce the concept of the probability of an event and then show how these probabilities can be computed in certain situations.
- Need the concept of the sample space and the events of an experiment.

2.2 Sample space and events

Sample space All possible outcomes of an experiment.

Some examples:

1. The sex of a newborn child: $S = \{g, b\}$
2. The order of finish in a race among 7 horses having post positions 1, 2, 3, 4, 5, 6, 7:

   $S = \{$all 7! permutations of (1, 2, 3, 4, 5, 6, 7)$\}$
3. The outcomes of flipping two coins:

$$S = \{(H, H), (H, T), (T, H), (T, T)\}$$

4. The outcomes of tossing two coins:

$$S = \{(i, j) : \ i, j = 1, 2, 3, 4, 5, 6\}$$

5. The lifetime of a transistor:

$$S = \{x : \ 0 \leq x < \infty\}$$

Event Any subset of the sample space.

Previous examples:

1. $E = \{g\}$
2. $E = \{$all outcomes in S starting with a $3\}$
3. $E = \{(H, H), (H, T)\}$
4. $E = \{(1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)\}$
5. $E = \{x : \ 0 \leq x \leq 5\}$

Operations on events

Union $E \cup F$: All points are either in E or in F or in both E and F.

Intersection $E \cap F$ (EF): All points are in both E and F.

Mutually exclusive If $E \cap F = \emptyset$.

Union of infinite events $\bigcup_{n=1}^{\infty} E_n$: All points are in E_n for at least one value of $n = 1, 2, \ldots$.

Intersection of infinite events $\bigcap_{n=1}^{\infty} E_n$: All points are in all events of $E_n, n = 1, 2, \ldots$.

Complement E^c: All points in the sample space S are not in E.

$$S^c = \emptyset$$

Contained $E \subset F$

Venn diagram A graphical representation is very useful for illustrating logical relations among events.

Rules of logical operations on events

Commutative	$E \cup F = F \cup E$	$EF = FE$
Associative	$(E \cup F) \cup G = E \cup (F \cup G)$	$(EF)G = E(FG)$
Distributive	$(E \cup F)G = EG \cup FG$	$EF \cup G = (E \cup G)(F \cup G)$

DeMorgan's laws:

$$\left(\bigcup_{i=1}^{n} E_i\right)^c = \bigcap_{i=1}^{n} E_i^c$$

$$\left(\bigcap_{i=1}^{n} E_i\right)^c = \bigcup_{i=1}^{n} E_i^c$$

2.3 Axioms of probability

For each event E of the sample space S, we define $n(E)$ to be the number of time in the first n repetitions of the experiment that the event E occurs.

$$P(E) = \lim_{n \to \infty} \frac{n(E)}{n}$$

Axiom 1

$$0 \leq P(E) \leq 1$$

Axiom 2

$$P(S) = 1$$

Axiom 3 For any sequence of mutually exclusive events $E_1, E_2, \ldots$ (that is, events for which $E_iE_j = \emptyset$ when $i \neq j$),

$$P\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} P(E_i)$$

We refer to $P(E)$ as the probability of the event E.

For any finite sequence of mutually exclusive events $E_1, E_2, \ldots, E_n$,

$$P\left(\bigcup_{1}^{n} E_i\right) = \sum_{i=1}^{n} P(E_i)$$

Example 2.3a. If our experiment consists of tossing a coin and if we assume that a head is as likely to appear as a tail, then we would have

$$P(\{H\}) = P(\{T\}) = \frac{1}{2}$$

- If the coin were biased and we felt that a head were twice as likely to appear as a tail, then we would have

$$P(\{H\}) = \frac{2}{3} \quad P(\{T\}) = \frac{1}{3}$$

Example 2.3b. If a die is rolled and we suppose that all six sides are equally likely to appear, then we would have $P(\{1\}) = P(\{2\}) = P(\{3\}) = P(\{4\}) = P(\{5\}) = P(\{6\}) = \frac{1}{6}$ From Axiom 3 it would thus follow that the probability of rolling an even number would equal

$$P(\{2,4,6\}) = P(\{2\}) + P(\{4\}) + P(\{6\}) = \frac{1}{2}$$

The assumption of the existence of a set function P, defined on the event of a sample space S, and satisfying Axioms 1, 2, and 3, constitutes the modern mathematical approach to probability theory.

2.4 Some simple propositions

- $1 = P(S) = P(E \cup E^c) = P(E) + P(E^c)$

Propositions 4.1

$$P(E^c) = 1 - P(E)$$

Propositions 4.2

$$\text{If } E \subset F, \text{ then } P(E) \leq P(F).$$

- Since $E \subset F$, then $F = E \cup E^cF$.
- From Axiom 3, $P(F) = P(E) + P(E^cF)$, which proves the result, since $P(E^cF) \geq 0$.

Proposition 4.3

$$P(E \cup F) = P(E) + P(F) - P(EF)$$

- From Axiom 3,

$$\begin{aligned} P(E \cup F) &= P(E \cup E^cF) \\ &= P(E) + P(E^cF) \end{aligned}$$

- Since $F = EF \cup E^cF$, we again obtain from Axiom 3 that

$$P(F) = P(EF) + P(E^cF)$$

thus completing the proof.

Example 2.4a. Suppose that we toss two coins and suppose that each of the four points in the sample space

$$S = \{(H,H),(H,T),(T,H),(T,T)\}$$

is equally likely and hence has probability $\frac{1}{4}$.

- Let $E = \{(H,H),(H,T)\}$ and $F = \{(H,H),(T,H)\}$.
- $$\begin{aligned} P(E \cup F) &= P(E) + P(F) - P(EF) \\ &= \frac{1}{2} + \frac{1}{2} - P(\{H,H\}) \\ &= 1 - \frac{1}{4} \\ &= \frac{3}{4} \end{aligned}$$

Probability of any one of the three events E or F or G occurs: $P(E \cup F \cup G) = P(E) + P(F) + P(G) - P(EF) - P(EG) - P(FG) + P(EFG)$

Proposition 4.4

$$P(E_1 \cup E_2 \cup \cdots \cup E_n) = \sum_{i=1}^{n} P(E_i) - \sum_{i_1<i_2} P(E_{i_1}E_{i_2}) + \cdots + (-1)^{r+1} \sum_{i_1<i_2<\cdots<i_r} P(E_{i_1}E_{i_2}\cdots E_{i_r}) + \cdots + (-1)^{n+1} P(E_1E_2\cdots E_n)$$

The summation $\sum_{i_1<i_2<\cdots<i_r} P(E_{i_1}E_{i_2}\cdots E_{i_r})$ is taken over all of the $\binom{n}{r}$ possible subsets of size r the set $\{1, 2, \ldots, n\}$.

2.5 Sample space having equally likely outcomes

- $S = \{1, 2, \ldots, N\}$
- $P(\{i\}) = \frac{1}{N}$
- $P(E) = \frac{\text{number of points in } E}{\text{number of points in } S}$

Example 2.5a. If two dice are rolled, what is the probability that the sum of the upturned faces will equal 7?

- $S = \{(i, j) |\ i, j = 1, 2, \ldots, 6\}$

- 6 possible outcomes:

$$(1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)$$

- The desired probability is $\frac{6}{36} = \frac{1}{6}$.

Example 2.5b. If 3 balls are "randomly drawn" from a bowl containing 6 white and 5 black balls, what is the probability that one of the drawn balls is white and the other two black?

- Regard the outcome of the experiment as the ordered set of drawn balls:
  - Sample space contains $11 \cdot 10 \cdot 9 = 990$ outcomes.
  - There are $6 \cdot 5 \cdot 4 = 120$ outcomes in which the first ball selected is white and the other two black.
  - $5 \cdot 6 \cdot 4 = 120$ outcomes in which the first is black, the second white and the third black.

– $5 \cdot 4 \cdot 6 = 120$ outcomes in which the first two are black, and the third two white.

– The desired probability is $\frac{120+120+120}{990} = \frac{4}{11}$.

- Regard the outcome of the experiment as the unordered set of drawn balls:

– $\binom{11}{3} = 165$ outcomes in S.

– $\binom{6}{1}\binom{5}{2} = 4$ desired outcomes.

– $\frac{\binom{6}{1}\binom{5}{2}}{\binom{11}{3}} = \frac{4}{11}$

Example 2.5c. A committee of 5 is to be selected from a group of 6 men and 9 women. If the selection is made randomly, what is the probability that the committee of 3 men and 2 women?

- The desired probability is $\frac{\binom{6}{3}\binom{9}{2}}{\binom{15}{5}} = \frac{240}{1001}$.

Example 2.5d. An urn contains n balls, of which one is special. If k of these balls are withdrawn one at a time, with each selection being equally likely to be any of the balls that remain at the time, what is the probability that the special ball is chosen?

- $P\{\text{special ball is selected}\} = \frac{\binom{1}{1}\binom{n-1}{k-1}}{\binom{n}{k}} = \frac{k}{n}$
- Alternative:
  - A_i : The special ball is the ith ball to be chosen, $i = 1, \ldots, k$.
  - $P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{k} P(A_i) = \dfrac{k}{n}$
  - $P(A_i) = \frac{(n-1)!}{n!} = \frac{1}{n}$

Example 2.5e. Suppose that $n+m$ balls, of which n are red and m are blue, are arranged in a linear order in such a way that all $(n+m)!$ possible orderings are equally likely. If we record the result of this experiment by only listing the colors of the successive balls, show that all the possible results remain equally likely.

- Every ordering of the colors has probability $\frac{n!m!}{(n+m)!}$ of occurring.
- 2 red balls: r_1, r_2; 2 blue balls: $b1_1, b_2$.
- The following orderings result in the successive balls alternating in color with a red ball first:

$$r_1, b_1, r_2, b_2 \quad r_1, b_2, r_2, b_1 \quad r_2, b_1, r_1, b_2 \quad r_2, b_2, r_1, b_1$$

- Each of the possible orderings of the colors has probability $\frac{4}{24} = \frac{1}{6}$.

Example 2.5f. A poker hand consists of 5 cards. If the cards have distinct consecutive value and are not all of the same suit, we say that the hand is a straight. For instance, a hand consisting of the five of spades, six of spades, seven of spades, eight of spades, and nine of hearts is a straight. What is the probability that one is dealt a straight?

- $\binom{52}{5}$ possible poker hands.
- 4^5 hands leading to exactly one ace, two, three, four, and five.

- $4^5 - 4$ hands make up a straight of the form ace, two, three, four, and five.
- $10(4^5 - 4)$ hands are straight.
- The desired probability: $\frac{10(4^5-4)}{\binom{52}{5}} \approx .0039$.

Example 2.5g. A 5-card poker hand is said to be a full house if it consist of 3 cards of the same denomination and 2 cards of the same denomination. What is the probability that one is dealt a full house?

- There are $\binom{52}{5}$ possible hands.
- There are $\binom{4}{2}\binom{4}{3}$ different combinations of, say 2 tens and 3 jacks.
- There are 13 different choices for the kind of pair and, after a pair has been chosen, there are 12 other choices for the denomination of the remaining 3 cards.

- The probability of a full house

$$\frac{13 \cdot 12 \cdot \binom{4}{2}\binom{4}{3}}{\binom{52}{5}} \approx .0014$$

Example 2.5h. In the game of bridge the entire deck of 52 cards is dealt out to 4 players. What is the probability that

(a) one of the players receives all 13 spades;

- There are $\binom{52}{13,13,13,13}$ possible divisions of the cards among the 4 distinct players.
- There are $\binom{39}{13,13,13}$ possible divisions of the cards leading to a fixed player having all 13 spades.
- The desired probability is $\frac{4\binom{39}{13,13,13}}{\binom{52}{13,13,13,13}} \approx 6.3 \times 10^{-12}$.

(b) each player receives 1 ace?

- There are $\binom{48}{12,12,12,12}$ possible divisions of the other 48 cards when each player is to receive 12.

– There are 4! ways of dividing the 4 aces so that each player receives.

– The desired probability is $\frac{4!\binom{48}{12,12,12,12}}{\binom{52}{13,13,13,13}} \approx$ .105.

Example 2.5i. It n people are present in a room, what is the probability that no two of them celebrate their birthday on the same day of the year? How large need n be so that this probability is less than $\frac{1}{2}$?

- There are 365^n possible outcomes.
- The desired probability is

$$p_n = (365)(364)\cdots(365-n+1)/(365)^n$$

- When $n \geq 23$, $p_n \leq \frac{1}{2}$.
- When $n = 50$, $1 - p_n = .970$.
- When $n = 100$, $1 - p_n \geq \frac{3\times10^6}{3\times10^6+1}$.

Example 2.5j. A deck of 52 playing cards is shuffled and the cards turned up one at a time until the first ace appears. Is the next card–that is, the card following the first ace–more likely to be the ace of spades or the two of clubs?

- There are 51! orderings of the ace of spades immediately following the first ace.
- There are 51! orderings of the two of clubs immediately following the first ace.
- $P\{$the ace of spades follows the first ace$\} = P\{$the two of club follows the first ace$\} = \frac{(51)!}{(52)!} = \frac{1}{52}$

Example 2.5k. A football team consists of 20 offensive and 20 defensive players. The player are to be paired in groups of 2 for the purpose of determining roommates. If the pairing is done at random, what is the probability that there are no offensive-defensive roommate pairs? What is the probability that there are $2i$ offensive-defensive roommate pairs, $i = 1, 2, \ldots, 10$?

- $\binom{40}{2,2,\ldots,2} = \frac{(40)!}{(2!)^{20}}$ ways of dividing the 40 players into 20 ordered pairs of two each.

- $$P_{2i} = \frac{\binom{20}{2i}^2 (2i)! \left[\frac{(20-2i)!}{2^{10-i}(10-i)!}\right]^2}{\frac{(40)!}{2^{20}(20)!}} \quad i = 0, 1, \ldots, 10$$

- Hence the probability of no offensive-defensive roommate pairs call it P_0, is given by

$$\begin{aligned} P_0 &= \frac{\left(\frac{(20)!}{2^{10}(10)!}\right)^2}{\frac{(40)!}{2^{20}(20)!}} = \frac{[(20)!]^3}{[(10)!]^2(40)!} \\ &\approx 1.3403 \times 10^{-6} \\ P_{10} &\approx .345861 \\ P_{20} &\approx 7.6068 \times 10^{-6} \end{aligned}$$

Next three examples illustrate the usefulness of Proposition 4.4.

Example 2.5l. A total of 36 members of a club play tennis, 28 play squash, and 18 play badminton. Furthermore, 22 of the members play both tennis and squash, 12 play both tennis and badminton, 9 play both squash and badminton. and 4 play all three sports. How many members of this club play at least one of these sports?

- $P(C) = \dfrac{\text{number of members in } C}{N}$
  N: The number of members of the club.
- T: The set of members that plays tennis.
  S: The set of members that plays squash.
  B: The set of members that plays badminton.

$$\begin{aligned} P(T \cup S \cup B) &= P(T) + P(S) + P(B) - P(TS) \\ &\quad - P(TB) - P(SB) + P(TSB) \\ &= \frac{36 + 28 + 18 - 22 - 12 - 9 + 4}{N} \\ &= \frac{43}{N} \end{aligned}$$

Example 2.5m. *The matching problem.*

Suppose that each of N men at a party throws his hat into the center of the room. The hates are first mixed up, and then each man randomly selects a hat. What is the probability that

(a) none of the men selects his own hat;

– E_i: ith man selects his own hat.

– $P\left(\bigcup_{i=1}^{N} E_i\right) = \sum_{k=1}^{N} (-1)^{k+1} \sum_{i_1<i_2<\cdots<i_k} P(E_{i_1} \cdots E_{i_k})$

– $1 - P\left(\bigcup_{i=1}^{N} E_i\right) = 1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^N}{N!}$

(b) exactly k of the men select their own hats?

– None of $N-k$ men selects his own hat: $1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^{N-k}}{(N-k)!}$

– $\binom{N}{k}$ possible selections of a group of k men.

– $\frac{\binom{N}{k}(N-k)!\left[1-1+\frac{1}{2!}-\frac{1}{3!}+\cdots+\frac{(-1)^{N-k}}{(N-k)!}\right]}{N!}$

$= \frac{1-1+\frac{1}{2!}-\frac{1}{3!}+\cdots+\frac{(-1)^{N-k}}{(N-k)!}}{k!}$

$$\approx \frac{e^{-1}}{k!}$$

Example 2.5n. If 10 married couples are seated at random at a round table, compute the probability that no next to her husband.

- E_i: ith couple sit next to each other.
- The desired probability is $1 - P\left(\bigcup_{i=1}^{n} E_i\right)$.
- $P(E_{i_1} E_{i_2} \cdots E_{i_n}) = \frac{2^n(19-n)!}{(19)!}$
- The probability that at least one married couple sits together equals $\binom{10}{1}2^1\frac{(18)!}{(19)!} - \binom{10}{2}2^2\frac{(17)!}{(19)!} + \binom{10}{3}2^3\frac{(16)!}{(19)!} - \cdots - \binom{10}{10}2^{10}\frac{(9)!}{(19)!} \approx .6605$
- The desired probability is approximately .3395.

***Example 2.5o. Runs** Consider an athletic team that had just finished its season with a final record of n wins and m losses. By examining the sequence of wins and losses, we are hoping to determine whether the team had stretches of games in which it was more likely to win than at other times. One way to gain insight into this question is to count the number of runs of wins and then see how likely that result would when all $(n+m)!/n!m!$ orderings of the n wins and m losses are assumed equally likely. By a run of wins we mean a consecutive sequence of wins. For instance, if $n = 10, m = 6$ and the sequence of outcomes was $WWWLLWWWLWLLLWWWW$, then there would be 4 runs of wins–the first run being of size 2, the second of size 3, the third of size 1, and the fourth of size 4.

- There are $\binom{n+m}{n}$ orderings are equally likely.
- Assume r runs of wins.
- x_i: The size of ith run.
- $x_1 + \cdots + x_r = n \qquad x_i > 0$
- y_i: The number of losses between $(i-1)$th runs of wins and ith runs of wins.

- $y_1 + \cdots + y_{r+1} = m \quad y_1, y_{r+1} \geq 0, y_i > 0$
- $\bar{y}_1 = y_1 + 1, \bar{y}_{r+1} = y_{r+1} + 1, \bar{y}_i = y_i$
- $\bar{y}_1 + \cdots + \bar{y}_{r+1} = m + 2 \qquad \bar{y}_i > 0$
- There are $\binom{m+1}{r}$ such outcomes.
- There are $\binom{n-1}{r-1}$ such outcomes for x_is.
- $P(\{r \text{ runs of wins}\}) = \frac{\binom{m+1}{r}\binom{n-1}{r-1}}{\binom{n+m}{n}}$
- If $n = 8, m = 6$, then the probability of 7 runs is $\frac{\binom{7}{7}\binom{7}{6}}{\binom{14}{8}} = 1/429$.
- Hence, if the outcome was $WLWLWLWLWWLWLW$, then we might suspect that the team's win probability was changing over time.
- On the other extreme, if the outcome were $WWWWWWWWLLLLLL$, then there would have been only 1 run, it would thus again seem unlikely that the team's win probability remained unchanged over its 14 games.

*2.6 Probability as a continuous set function

If $\{E_n, n \geq 1\}$ is an increasing (decreasing) sequence of event, then we define a new event. denoted by $\lim_{n\to\infty} E_n$, by

$$\lim_{n\to\infty} E_n = \bigcup_{i=1}^{\infty} E_i \quad \left(\bigcap_{i=1}^{\infty} E_i\right)$$

Proposition 2.6.1 If $\{E_n, n \geq 1\}$ is either an increasing or a decreasing sequence of events, then

$$\lim_{n\to\infty} P(E_n) = P(\lim_{n\to\infty} E_n)$$

- Suppose $\{E_n, n \geq 1\}$ is an increasing sequence and define the events $F_n, n \geq 1$ by
- $F_1 = E_1$
- $F_n = E_n \left(\bigcup_{i=1}^{n-1} E_i\right)^c = E_n E_{n-1}^c \quad n > 1$
- Used $\bigcup_{i=1}^{n-1} E_i = E_{n-1}$

- So $\overset{\infty}{\underset{i=1}{\cup}} F_i = \overset{\infty}{\underset{i=1}{\cup}} E_i$ and $\overset{n}{\underset{i=1}{\cup}} F_i = \overset{n}{\underset{i=1}{\cup}} E_i$
-

$$
\begin{aligned}
P\left(\overset{\infty}{\underset{1}{\cup}} E_i\right) &= P\left(\overset{\infty}{\underset{1}{\cup}} F_i\right) \\
&= \overset{\infty}{\underset{1}{\Sigma}} P(F_i) \quad \text{(by Axiom 3)} \\
&= \lim_{n\to\infty} \overset{n}{\underset{1}{\Sigma}} P(F_i) \\
&= \lim_{n\to\infty} P\left(\overset{n}{\underset{1}{\cup}} F_i\right) \\
&= \lim_{n\to\infty} P\left(\overset{n}{\underset{1}{\cup}} E_i\right) \\
&= \lim_{n\to\infty} P(E_n)
\end{aligned}
$$

which proves the result when $\{E_n, n \geq 1\}$ is increasing.

- The proof for decreasing events is similar.

Example 2.6a. *Probability and a paradox.*

Suppose that we posses an infinitely large urn and an infinite collection of balls labeled ball number 1, number 2, number 3, and so on. Consider an experiment performed as follows.

At 1 minute to 12 P.M., balls numbered 1 through 10 are placed in the urn, and ball number 10 is withdrawn. At $\frac{1}{2}$ minute to 12 P.M., balls numbered 11 through 20 are placed in the urn, and ball number 20 is withdrawn. At $\frac{1}{4}$ minute to 12 P.M., balls numbered 21 through 30 are placed in the urn, and ball number 30 is withdrawn. At $\frac{1}{8}$ minute to 12 P.M., and so on. The question of interest is, how many balls are in the urn at 12 P.M.?

- There is an infinite number of balls in the urn at 12 P.M.
- Let us change the experiment and suppose that at 1 minute to 12 P.M. balls numbered 1 through 10 are placed in the urn, and ball number 1 is withdrawn. At $\frac{1}{2}$ minute to 12 P.M., balls numbered 11 through 20 are placed in the urn, and ball number 2 is withdrawn. At $\frac{1}{4}$ minute to 12 P.M., balls numbered 21 through 30 are placed in the urn, and ball number 3 is withdrawn. At $\frac{1}{8}$

minute to 12 P.M., and so on.

– The urn is empty at 12 P.M.

- Let us now suppose that whenever a ball is to be withdrawn that ball is randomly selected from among those present.

  – We shall show that, with probability 1, the urn is empty at 12 P.M.

  – E_n: The event the event that ball number 1 is still in the urn after the first n withdrawals have been made.

  – $P(E_n) = \frac{9 \cdot 18 \cdot 27 \cdots (9n)}{10 \cdot 19 \cdot 28 \cdots (9n+1)}$

  – $P\{$ball number 1 is still in the urn at 12 P.M.$\}$
  $P\left(\bigcap_{i=1}^{n} E_n\right) = \lim_{n\to\infty} P(E_n) = \prod_{i=1}^{\infty}\left(\frac{9n}{9n+1}\right)$

  – $\prod_{i=1}^{\infty}\left(\frac{9n+1}{9n}\right) = \prod_{i=1}^{\infty}\left(1+\frac{1}{9n}\right) = \infty$

  – Hence, let F_i denote the event that ball number i is in the urn at 12 P.M., we can show similarly $P(F_i) = 0$.

2.7 Probability as a measure of belief

Example 2.7a. Suppose that in a 7-horse race you feel that each of the first 2 horses has a 20 percent chance of winning, horses 3 and 4 each has a 15 percent chance, and the remaining 3 horses, a 10 percent chance each. Would it be better for you to wager at even money, that the winner will be one of the first three horses, or to wager, again at even money, that the winner will be one of the horses 1,5,6,7?

- The probability of winning the first bet is $.2 + .2 + .15 = .55$
- It is $.2 + .1 + .1 + .1 = .5$ for the second.
- Hence the first wager is more attractive.

Summary

- Sample space S: The set of all possible outcomes of a an experiment.
- $\bigcup_{i=1}^{n} A_i$: All outcomes that are in at least one of the events.

- $\bigcap\limits_{i=1}^{n} A_i$: All outcomes that are in all of the events.
- A^c: All outcomes that are not in A.
- $\emptyset$: The null set.
- Mutually exclusive: $AB = \emptyset$
- Axiom of probability:

 (i) $0 \leq P(A) \leq 1$

 (ii) $P(S) = 1$

 (iii) For mutually exclusive sets a_i

$$P\left(\bigcup\limits_{i=1}^{n} A_i\right) = \sum\limits_{i=1}^{n} P(A_i)$$

- $P(A^c) = 1 - P(A)$
- $P(A \cup B) = P(A) + P(B) - P(AB)$
- $P\left(\bigcup\limits_{i=1}^{n} A_i\right) = \sum\limits_{i_1 < i_2 < \cdots < i_k} (-1)^{k+1} P(A_{i_1} A_{i_2} \cdots A_{i_k})$
- If S is finite and each one point set is assumed to have equal probability, then

$$P(A) = \frac{|A|}{|S|}$$