

Lesson 1. Combinatorial Analysis, Probability Spaces

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- 1. Permutations.** How many different orders of n objects exist?
 - Without repetition (there are n different objects): $n!$
 - With repetition (there are n objects of which n_1, \dots, n_r are alike):
 $\frac{n!}{n_1! \dots n_r!}$
- 2. Variations.** How many different orders of k objects selected from a set of n objects exist?
 - Without repetition (an object is selected at most once): $n(n-1) \dots (n-k+1) = \frac{n!}{(n-k)!}$
 - With repetition (an object may be selected several times): n^k
- 3. Combinations.** How many different ways k objects can be selected from a set of n objects?
 - Without repetition (an object is selected at most once): $\frac{n!}{k!(n-k)!} = \binom{n}{k}$, $k \leq n$
 - With repetition (an object may be selected several times): $\binom{k+n-1}{k-1}$, $k \in \mathbb{N}$

Identities containing binomial coefficients:

$$(a+b)^n = \sum_{k=0}^n \binom{n}{k} a^k b^{n-k} \quad 2^n = \sum_{k=0}^n \binom{n}{k} \quad 0 = \sum_{k=0}^n (-1)^k \binom{n}{k}$$
$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k} \quad \binom{n+m}{k} = \sum_{i=0}^k \binom{n}{i} \binom{m}{k-i} \quad \text{if } k \leq \min\{n, m\}$$

Probability Space

$(\mathcal{S}, \mathcal{A}, \mathbb{P})$, where \mathcal{S} is the *sample space* (set of all possible outcomes=*elementary events*), $\mathcal{A} = \{A \mid A \subset \mathcal{S}\}$ is the Boole-algebra of the all possible events (including \emptyset =impossible/null event and \mathcal{S} =certain/sure event), and the set function $\mathbb{P} : \mathcal{A} \rightarrow \mathbb{R}$ satisfies the following **AXIOMS**:

1. For any $A \in \mathcal{A}$: $0 \leq \mathbb{P}(A) \leq 1$
2. $\mathbb{P}(\mathcal{S}) = 1$
3. For any sequence of mutually exclusive events A_1, A_2, \dots : $\mathbb{P}(\sum_i A_i) = \sum_i \mathbb{P}(A_i)$

Propositions implied by the axioms:

- $\mathbb{P}(\bar{A}) = 1 - \mathbb{P}(A)$
- Probability is a monotonous set function: if $A \subset B$, then $\mathbb{P}(A) \leq \mathbb{P}(B)$.
- $\mathbb{P}(\sum_{i=1}^n A_i) = \sum_{k=1}^n (-1)^{k-1} S_k$, $S_k = \sum_{1 \leq i_1 < i_2 < \dots < i_k \leq n} \mathbb{P}(A_{i_1} A_{i_2} \dots A_{i_k})$
(inclusion-exclusion)

De Morgan identities:

$$\overline{\sum_{i=1}^n A_i} = \prod_{i=1}^n \bar{A}_i \quad \text{and} \quad \overline{\prod_{i=1}^n A_i} = \sum_{i=1}^n \bar{A}_i$$

Examples of probability spaces:

- *Combinatorial*: the sample space has finite number of equally like outcomes, $\mathbb{P}(A) = |A|/|\mathcal{S}|$.
- *Geometric*: the sample space is a region with finite measure μ (length, area, volume), $\mathbb{P}(A) = \mu(A)/\mu(\mathcal{S})$.