Exercises to Conditional Probability, Independence, and Inclusion–Exclusion

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1. *Matching problem*: Suppose that each of n students brings a present to a party. The presents are mixed up and then each student randomly selects a present. What is the probability that none of the students selects his/her own present? (Equivalent problem: n married couples are dancing: n man–woman pairs are formed randomly. What is the probability that none of the men selects his own wife?)

2. How many fair coins should be tossed so that the probability of having at least one head be more than 0.9?

3. By my knowledge, the neighboring family with two children has at least one boy. On this condition, what is the probability that both children are boys?

4. There are 3 pubs in our little town and I am looking for my best friend, who is in pub with probability 60%. I have not found him in the first pub, and I have not found him in the second one. On this condition, what is the probability that I shall find him in the third pub?

5. Mosquitoes are sprayed in three steps. In the first step, they get rid of 70%, in the second step they get rid of 50%, and in the third step they get rid of 30% of the existing mosquitoes. What is the probability that a mosquito survives all the three sprayings?

6. 0.1% of the population suffers of a certain illness. A medical test for it works with error: sometimes it diagnoses a healthy person ill, or an ill person healthy; both kinds of error happen with probability 0.01. Having a positive test result (indicating illness), what is the probability

that I am still healthy? Having two positive results (of two independent medical tests), what is the probability that I am still healthy?

7. **Secretary Problem** (Marriage or Sultan's dowry problem). Secretaries are interviewed one after the other in the following way:

   - There is one secretarial position available.
   - The number n of applicants is known.
   - The applicants are interviewed sequentially in random order, each order being equally likely.
   - It is assumed that you can rank all the applicants from best to worst without ties. The decision to accept or reject an applicant must be based only on the relative ranks of those applicants interviewed so far.
   - An applicant once rejected cannot later be recalled.

   What is the probability that you find the best one by the following m-strategy: for some integer $0 < m < n$ you reject the first m applicants, and then choose the next applicant who is best in the relative ranking of the observed applicants. What is the optimal m? (It can be shown that this is the best strategy.)

   Find the limit of the $\frac{m}{n}$ ratio at the passage to infinity. Prove that it is $\frac{1}{e} \approx 0.37$, so 37% can let go.