Lesson 1. Conditional Probability, Bayes Rule, Independence

Marianna Bolla, Prof, DSc.
Institute of Mathematics, BME

December 19, 2020

1 Conditional Probability and Implied Theorems

- *Definition.* $\mathbb{P}(A|B) = \frac{\mathbb{P}(AB)}{\mathbb{P}(B)}, \quad \mathbb{P}(B) > 0$. With B fixed, $\mathbb{Q}(A) := \mathbb{P}(A|B)$. $(\mathcal{S}, \mathcal{A}, \mathbb{Q})$ is also a probability space with all of its consequences.

- *Definition.* $B_1, B_2, \dots$ is a complete set of mutually exclusive (disjoint) events, if $B_i B_j = \emptyset$ $(i \neq j)$ and $\sum_i \mathbb{P}(B_i) = 1$.

- **Theorem** (of complete probability). Let $B_1, B_2, \dots$ be a complete set of mutually exclusive events and A be an arbitrary event. Then

$$\mathbb{P}(A) = \sum_i \mathbb{P}(A|B_i) \cdot \mathbb{P}(B_i).$$

- **Bayes' Theorem.** Let $B_1, B_2, \dots$ be a complete set of mutually exclusive events and A be an arbitrary event. Then

$$\mathbb{P}(B_k|A) = \frac{\mathbb{P}(A|B_k) \cdot \mathbb{P}(B_k)}{\sum_i \mathbb{P}(A|B_i) \cdot \mathbb{P}(B_i)}, \qquad k = 1, 2, \dots$$

- **Theorem** (factorization). Let $A_1, A_2, \dots, A_n$ be arbitrary events. Then

$$\mathbb{P}(A_1 A_2 \dots A_n) = \mathbb{P}(A_1) \cdot \mathbb{P}(A_2|A_1) \dots \mathbb{P}(A_n|A_1 \dots A_{n-1}).$$

Independence

- *Definition*. A and B are independent if

$$\mathbb{P}(AB) = \mathbb{P}(A) \cdot \mathbb{P}(B).$$

  Remark: if $\mathbb{P}(A) \neq 0$ and $\mathbb{P}(B) \neq 0$, then A and B cannot be exclusive and independent at the same time. $\mathcal{S}$ and $\emptyset$ are independent of any other event.

- *Definition*. The events $A_1, \ldots, A_n$ are (completely) independent if

$$\mathbb{P}(A_1 \ldots A_n) = \mathbb{P}(A_1) \ldots \mathbb{P}(A_n).$$

  (If $\mathbb{P}(A_i A_j) = \mathbb{P}(A_i)\mathbb{P}(A_j)$ for $i \neq j$, then $A_1 \ldots A_n$ are pairwise independent; this is weaker than independence.)