Causal Vector Autoregression (advisor: *Marianna Bolla*)

Problem description

The task is to test a novel model connecting causality and autoregession in multivariate time series. Above completing the theoretical proofs (I have done parts of those) the research also includes technical issues: implementation an algorithm using block matrix techniques and application to simulated and real life data.

The applicants must have some maturity in probability, statistics, and matrix analysis. Knowledge of stochastic processes, time series, and graphical models is not necessary, I will give the basics in the first lesson and some definitions below. Applicants having good programming skills (e.g. in Python) and ready to implement matrix decomposition algorithms, are welcome.

The discrete time time series $\{X_t\}$ $(t = 0, \pm 1, \pm 2, \dots)$ is called weakly stationary (or stationary in the wide sense) if X_ts have the same expectation and the covariance function $c(h) = cov(X_{t+h}, X_t)$ for $h = 0 \pm 1, \pm 2, \dots$ does not depend on the time t. Note that in case of a real state space (X_ts take on real values, in many cases, they are Gaussian), $c(-h) = c(h)$. The notion naturally extends to multi-dimension via cross-covariance matrices.

Now the d-dimensional, weakly stationary time series $\{\mathbf{X}_t\}$ obeys the proposed model for causal autoregression:

$$\boldsymbol{A}\mathbf{X}_t + \boldsymbol{B}\mathbf{X}_{t-1} = \mathbf{U}_t, \quad t = 1, 2, \dots$$

where the white noise random vector $\mathbf{U}_t$ is uncorrelated with $\mathbf{X}_{t-1}$, has zero expectation and diagonal covariance matrix, $\boldsymbol{B}$ is a $d \times d$ matrix, and $\boldsymbol{A}$ is $d \times d$ upper triangular with 1's along its main diagonal. The model can be transformed into a first order autoregression and it gives an alternative solution to the famous Yule-Walker equations. In this form, via the matrix $\boldsymbol{A}$, the model expresses contemporaneous causation between the components of $\mathbf{X}_t$ if they have a topological ordering of a DAG (Directed Acyclic Graph), akin to the recursive graphical models. In the decomposable case, $\boldsymbol{A}$ has a so-called reducible zero pattern that can be concluded from the autocovariances. This issue, together with higher order autoregressions is to be investigated.

For the solution, block matrix decompositions, like the LDL or block Cholesky can be used. The recursion and adaptation of these algorithms to our problem is also a challenge.

Qualifying exercises

1. Consider the symmetric, one-dimensional random walk defined by

$$X_0 = 0, \quad X_t = \sum_{j=1}^{t} \xi_j, \quad t = 1, 2, \dots,$$

where $\{\xi_j\}$ is a binary process: ξ_js are i.i.d. (independent, identically distributed) and $\xi_j = \pm 1$ with probability $\frac{1}{2} - \frac{1}{2}$.

Prove that the random walk process $\{X_t\}$ is not weakly stationary.

2. Show that the one-dimensional process

$$X_t = A\cos(\omega t) + B\sin(\omega t), \quad t = 0, \pm 1, \pm 2, \dots$$

is weakly stationary, where A, B are uncorrelated, standard normal (Gaussian) random variables and $\omega \in [0, 2\pi)$ is a fixed frequency. Find the autocovariance function of the process too.

3. We have a DAG (Directed Acyclic Graph), in which there are no directed cycles. Prove that there is a topological ordering/labeling of the vertices such that for every $i \to j$ edge, the relation $j < i$ holds. (In a directed graphical model, the nodes correspond to random variables, and the $i \to j$ relation means that the random variable corresponding to i is the parent/cause of the one corresponding to j, so one can as well think of labels as ages.)