PROBABILITY AND STATISTICS, Problems to Lesson 5.

Definition: The **covariance** between X and Y (having finite second moments) is

$$cov(X,Y) = \mathbb{E}(X - \mathbb{E}X)(Y - \mathbb{E}Y) = \mathbb{E}(XY) - \mathbb{E}(X) \cdot \mathbb{E}(Y),$$

while their **correlation** is

$$corr(X,Y) = \frac{cov(X,Y)}{\mathbb{D}(X) \cdot \mathbb{D}(Y)}.$$

By the Cauchy–Schwarz inequality: $|corr(X,Y)| \leq 1$.

1. *Zeta (Zipf) distribution:* The p.m.f. of X is:

$$\mathbb{P}(X = k) = \frac{C}{k^{\alpha+1}}, \qquad k = 1, 2, \ldots$$

for some parameter $\alpha > 0$. Give C by means of the Riemann zeta function:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \qquad (s > 1).$$

Give $\mathbb{E}(X)$ and $\mathbb{D}(X)$ by means of the Riemann zeta function, if they exist.

2. The p.d.f. of an absolutely continuos random variable X is $f(x) = \sin x$, if $0 < x < \pi/2$, and 0, otherwise. Give the c.d.f. $F(x)$ of X. Calculate $\mathbb{P}(X > \pi/4)$ and $\mathbb{P}(X > \pi/4 \,|X < \pi/3)$. Find the median and expectation of X.

3. The joint p.d.f. of the random vector (X,Y) is

$$f(x,y) = x + y, \qquad if \quad 0 < x, y < 1$$

and 0, otherwise. Find the marginal distributions. Are X and Y independent rv's? Find the conditional distribution of Y given $X = x$, $\mathbb{P}(Y < 1/2|X = 1/4)$=?, $\mathbb{E}(Y|X)$=?

4. *Rendezvous problem*: Mary and John are to meet each other sometimes between 7 and 8 P.M. at a given place, and any of them waits for the other at most 20 minutes. What is the probability, that they meet each other, if they arrive randomly and independently to the given place within the (7,8) interval?

5. The random vector (X,Y) is uniformly distributed over the region bounded by the unit circle. Find the marginal distributions. Are X and Y independent rv's? Find the covariance between X and Y!

6. We have N balls of k different colors (n_1 red, n_2 white, $\ldots$,n_k of the kth color), mixed in an urn. n balls are selected at random without replacement (or at the same time). Suppose that $n \leq \min\{n_1, \ldots, n_k\}$. Let $\mathbf{X} = (X_1, \ldots, X_k)$ be the random vector with components X_i: number of the ith color balls among the n ones ($i = 1, \ldots, k$). Describe the distribution of $\mathbf{X}$ (*polyhypergeometric distribution*), and the marginal distribution of its components.

7. We have N balls of k different colors (n_1 red, n_2 white, $\ldots$,n_k of the kth color), mixed in an urn. n balls are selected at random with replacement. Let $\mathbf{X} = (X_1, \ldots, X_k)$ be the random vector with components X_i: number of the ith color balls among the n ones ($i = 1, \ldots, k$). Describe the distribution of $\mathbf{X}$ (*polynomial/multinomial distribution*), and the marginal distribution of its components.