PROBABILITY AND STATISTICS, Problems to Lesson 6.

1. Let $X \sim \mathcal{U}(-\pi/2, \pi/2)$. Prove that $Y = \tan(X)$ has standard Cauchy distribution (with no finite first moment).

2. Let X be exponentially distributed with parameter λ. Find the distribution of $Y = cX$, where c is a positive constant.

3. Let $X \sim \mathcal{N}(\mu, \sigma^2)$. Prove that $Y = \frac{X-\mu}{\sigma}$ has standard Gaussian distribution.

4. Let X be an absolutely continuous r.v. with c.d.f. $F(x)$. Prove that $F(X) \sim \mathcal{U}(0,1)$.

5. The random number generator approximately provides i.i.d. $\mathcal{U}(0,1)$ r.v.'s. Using this fact, write a computer program that generates i.i.d. exponentially distributed r.v.'s with a given positive parameter λ.

6. Let $X \sim \mathcal{B}_n(p)$ and $Y \sim \mathcal{B}_m(p)$ be independent r.v.'s. Prove that $X + Y \sim \mathcal{B}_{n+m}(p)$.

7. Let $X \sim \mathcal{P}(\lambda)$ and $Y \sim \mathcal{P}(\nu)$ be independent r.v.'s. Prove that $X + Y \sim \mathcal{P}(\lambda + \nu)$.

8. Let $X_1, \dots, X_n$ be i.i.d. exponentially distributed r.v.'s with parameter λ. Prove that $\sum_{i=1}^n X_i$ has Gamma-distrinbution with parameters n and λ. Hint: first apply the convolution formula for $n = 2$, and use induction.

9. Let $X, Y \sim \mathcal{U}(0,1)$ be independent r.v.'s. Find the distribution (c.d.f. and p.d.f.) of $X + Y$, XY, and X/Y.

10. Let the random vector (X, Y) have a two-variate normal (Gaussian) distribution. It is uniquely defined by the following parameters:

$$\mathbb{E}(X) = \mu_1, \quad \mathbb{E}(Y) = \mu_2, \quad \mathbb{D}^2(X) = \sigma_1^2, \quad \mathbb{D}^2(Y) = \sigma_2^2, \quad cov(X,Y) = c.$$

    Prove that for any real numbers a, b the distribution of $aX + bY$ is also Gaussian. Find its parameters! Find the marginal distributions too.

11. A fair dice is casted independently 600 times. Give an approximation (by CLT) for the probability that the number of 6's falls between 95 and 110.

12. *Elections 2000.* In a state (call it Florida) the voters vote for two candidates (with 0.5-0.5 probability), independently of each other. If there are 5 million voters, what is the probability that the difference of the votes given for the two candidates is less than 300 in absolute value.

13. 1000 person arrive to the left or right entrance of a theatre independently (they choose between the entrances with 0.5-0.5 probability). How many hangers to place into the left and right cloak rooms, if they want to give only a 1 percent chance to the event that somebody cannot place his/her coat in the nearest cloakroom. (Each pearson has one coat.)

14. Use the Weak Law of Large Numbers to prove that for the true probability p of an event and for its relative frequency $\overline{X}_n$ based on an n-element i.i.d. Bernoulli sample:

$$\mathbb{P}(|\overline{X}_n - p| \geq \varepsilon) \leq \frac{p(1-p)}{n\varepsilon^2} \leq \frac{1}{4n\varepsilon^2}, \qquad \forall \varepsilon > 0.$$

15. *Opinion poll.* On the basis of the previous exercise find the number n of people to be interviewed so that the true (but unknown) population support p of a candidate and its relative frequency based on this poll differ at most 0.01 with probability at least 90 percent.