POBABILITY AND STATISTICS, Problems to Lessons 9-10.

1. Let $X_1, \dots, X_n \sim \mathcal{P}(\lambda)$ be i.i.d. sample. Find sufficient statistic and ML-estimate of the parameter λ! Is the ML-estimate unbiased?

2. Let $X_1, \dots, X_n \sim \mathcal{E}xp(\lambda)$ be i.i.d. sample. Find sufficient statistic and ML-estimate of the parameter λ! Is the ML-estimate unbiased or asymptotically unbiased?

3. Let $X_1, \dots, X_n \sim \mathcal{N}(\mu, \sigma^2)$ be i.i.d. sample. Find sufficient statistic and ML-estimate of the parameter $\theta = (\mu, \sigma^2)$! Are the above estimators unbiased or asymptotically unbiased?

4. Let $X_1, \dots, X_n \sim \mathcal{U}[a, b]$ be i.i.d. sample! Find sufficient statistic and ML-estimate of the parameter $\theta = (a, b)$! Are the above estimates unbiased or asymptotically unbiased?

5. Prove that $\bar{X}$ gives an unbiased estimate of $\psi(\theta) = \mathbb{E}_\theta(X)$, whenever it exists.

6. Let $X_1, \dots, X_n \sim \mathcal{I}(\theta)$ be i.i.d. Bernoulli sample. Prove that $\bar{X}$ is an unbiased estimator of θ.

7. Let $X_1, \dots, X_n \sim \mathcal{E}xp(\lambda)$ be i.i.d. sample. Prove that $\bar{X}$ is an unbiased estimator of $1/\lambda$.

8. Prove that $\sum_{i=1}^n a_i X_i$ (with $\sum_{i=1}^n a_i = 1$) is also an unbiased estimator of $\psi(\theta) = \mathbb{E}_\theta(X)$, whenever it exists. Prove that the sample mean is the most efficient among the linear unbiased estimators of the expectation, whenever it exists.

9. Let $X_1, \dots, X_n \sim \mathcal{U}(0, \theta)$ be i.i.d. sample, $n \geq 2$! Prove that the non-linear estimator $\frac{n+1}{2n} X_n^*$ is an unbiased estimator of the expectation $\psi(\theta) = \theta/2$, and it is more efficient than $\bar{X}$.

10. Prove that the the sample mean is a strongly consistent estimator of the true expectation, if it exists. Prove that the the sample variance (and the corrected sample variance) are both strongly consistent estimators of the true variance, if it exists. Prove that the the sample covariance (and the corrected sample covariance) are both strongly consistent estimators of the true covariance, if it exists.

11. Let $X_1, \dots, X_n \sim \mathcal{I}(\theta)$ be i.i.d. Bernoulli sample. Find $I_n(\theta)$ and apply the Cramer–Rao inequality for the unbiased estimator $\bar{X}$ of θ.

12. Let $X_1, \dots, X_n \sim \mathcal{N}(\theta, \sigma_0^2)$ be i.i.d. Gaussian sample (with known variance). Find $I_n(\theta)$ and apply the Cramer–Rao inequality for the unbiased estimator $\bar{X}$ of θ.

13. Let $X_1, \dots, X_n \sim \mathcal{P}(\lambda)$ be i.i.d. sample. $T = \sum_{i=1}^n X_i$ is a sufficient statistic, and $S = X_1$ is an unbiased estimator for λ. Perform the blackwellization procedure with them!

14. Let $X_1, \dots, X_n \sim \mathcal{N}(\mu, \sigma^2)$ be i.i.d. Gaussian sample. Find moment estimators of the parameters μ, σ^2!

15. Let $X_1, \dots, X_n \sim \mathcal{U}(a, b)$ be i.i.d. sample! Find moment estimators of the parameters a, b!

16. Let $X_1, \dots, X_n \sim \mathcal{N}(\mu, \sigma_0^2)$ be i.i.d. Gaussian sample (with known variance). Construct a 95% confidence interval for μ, symmetric around $\bar{X}$!

17. Let $X_1, \dots, X_n \sim \mathcal{N}(\mu, \sigma^2)$ be i.i.d. Gaussian sample (with unknown variance). Construct a 95% confidence interval for μ, symmetric around $\bar{X}$! Use the following theorem.

    Lukács's Theorem: In the above setup, $\bar{X} \sim \mathcal{N}(\mu, \frac{\sigma^2}{n})$, $nS_n^2/\sigma^2 = (n-1)S_n^{*2}/\sigma^2 \sim \chi^2(n-1)$, and they are independent statistics.